Semialgebraic Outer Approximations for Set-Valued Nonlinear Filtering ⋆

Dario Piga * **Alessio Benavoli** **

* *IDSIA Dalle Molle Institute for Artificial Intelligence SUPSI-USI, Manno, Switzerland. (dario.piga@supsi.ch).*
** *Department of Computer Science and Information Systems (CSIS), University of Limerick, Ireland. (alessio.benavoli@ul.ie).*

Abstract: This paper addresses the set-valued filtering problem for discrete time-varying dynamical systems, whose process and measurement equations are polynomial functions of the system state. According to a set-membership framework, the process and measurement noises, as well as the initial state, are assumed to belong to bounded uncertainty regions, which are supposed to be generic semialgebraic sets described by polynomial inequalities. A sequential algorithm, based on sum-of-squares (SOS) representation of positive polynomials is proposed to compute a semialgebraic set described by an a-priori fixed number of polynomial constraints which is guaranteed to contain the true state of the system with certainty.

Keywords: Filtering; polynomial optimization; bounded error; semialgebraic sets; sum-of-square polynomials.

1. INTRODUCTION

The aim of this paper is to derive a *flexible, accurate* and *efficient* set-valued filtering algorithm for time-varying dynamical systems characterised by *polynomial nonlinearities.*

Its *flexibility* comes from the possibility for the practitioner to choose *any semialgebraic set* to outer approximate the set which is guaranteed to include the state values of the system with certainty. Outer approximating regions previously proposed in set-valued filtering, such as bounding boxes, ellipsoidal regions, parallelotopes and zonotopes can therefore be seen as particular cases of our approach. We stress the fact that our algorithm can be applied to any *nonlinear* dynamical system characterised by polynomial nonlinearities and, therefore, provides a flexible way to extend set-valued filtering based on bounding boxes, ellipsoidal regions, parallelotopes and zonotopes to such nonlinear systems.

Its *accuracy* comes from the formulation of this outer approximation problem as a procedure that *sequentially* minimises a *numerical* estimate of the volume of the outer-bounding set. The resulting membership set is therefore tight.

Its *efficiency* comes from the use of a *sum-of-squares relaxation* to formulate each iteration of our algorithm as a convex *semi-definite programming* (SDP) problem.

We prove theoretically that, despite these approximations, our membership set always includes the state values of the system with certainty.

Flexibility, accuracy and efficiency have been the main drivers in the development of algorithms for set-membership estimation (Milanese and Vicino, 1991; Combettes, 1993; Milanese and Novara, 2011; Casini et al., 2014; Cerone et al., 2014). Set-membership filtering was first proposed in (Schweppe, 1967; Bertsekas and Rhodes, 1971), where an ellipsoidal bounding of the state of linear dynamical systems is computed. The use of ellipsoidal sets to the state estimation problem has also been considered by other authors, for example (Kuntsevich and Lychak, 1992; Deller et al., 1994; Savkin and Petersen, 1998). In order to improve the estimation accuracy, convex polytopes instead of ellipsoids have been proposed in (Piet-Lahanier and Walter, 1989; Mo and Norton, 1990). Unfortunately such a polytope may be extremely complex (the number of vertices of the polytope increases exponentially in time) and the corresponding polytopic updating algorithms may require an excessive amount of calculations and storage. For this reason, the research effort in set-membership has then focused on efficient solutions. The idea to outer approximate the true polytope with a simpler polytope, *i.e.*, possessing a limited number of vertices (or, equivalently, faces), was considered in (Broman and Shensa, 1990). In this respect, a parallelotopic approximation of the set-membership set was presented in (Chisci et al., 1996, 1998). Minimum-volume bounding parallelotopes are then used to estimate the state of a discrete-linear dynamical system with polynomial complexity. Zonotopes have been proposed to reduce the conservativeness of parallelotopes. Zonotopes were used in (Puig et al., 2003; Combastel, 2003; Le et al., 2011) to build a state bounding observer in the context of linear discrete systems and in (Alamo et al., 2003) for nonlinear discrete-time systems. Similar approaches for set-membership estimation for nonlinear systems are presented in (Calafiore, 2005; El Ghaoui and Calafiore, 2001; Maier and Allgöwer, 2009), where ellip-

⋆ This work was partially supported by the H2020-723248 project *DAEDALUS - Distributed control and simulation platform to support an ecosystem of digital automation developers and by the Swiss NRP 75 Big Data grant no. 407540-167199.*

soids are used instead of zonotopes. Randomized methods are employed in (Dabbene et al., 2015) to approximate, with probabilistic guarantees, the uncertain state trajectory with polynomial sublevel sets. In (Benavoli and Piga, 2016), by exploiting results on filtering with sets of probability measures (Benavoli et al., 2009, 2011; Benavoli, 2013), it is showed that set-membership estimation can be equivalently formulated in the probabilistic setting by employing sets of probability measures. Moreover, as an application of the approach, a set-membership filter for polynomial nonlinear dynamical system based on non-symmetrical polytopic bounds is derived. The present work generalises this latter idea by allowing the practitioner to choose any semialgebraic set as membership set.

2. PROBLEM FORMULATION

Let us consider a discrete time-varying nonlinear dynamical system described by the difference equations:

$$\begin{cases} \mathbf{x}(k) = \mathbf{a}(\mathbf{x}(k-1), k-1) + \mathbf{w}(k-1), \\ \mathbf{y}(k) = \mathbf{c}(\mathbf{x}(k), k) + \mathbf{v}(k), \end{cases} \tag{1}$$

where $\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]^\top \in \mathbb{R}^n$ represent the state of the system at time k, $\mathbf{y}(k) \in \mathbb{R}^m$ is the vector of output measurements, $\mathbf{w}(k) \in \mathbb{R}^n$ and $\mathbf{v}(k) \in \mathbb{R}^m$ are the process and the measurement noise, respectively. The terms $\mathbf{a}(\mathbf{x}(k), k)$ and $\mathbf{c}(\mathbf{x}(k), k)$ are assumed to be polynomial functions in the state $\mathbf{x}(k)$ with known time-varying coefficients. [1]

According to a set-membership filtering framework, the only available information on the noises $\mathbf{w}(k), \mathbf{v}(k)$ and the initial state $\mathbf{x}(0)$ is that they belong to compact uncertainty sets, *i.e.*,

$$\mathbf{w}(k) \in \mathcal{W}_k, \quad \mathbf{v}(k) \in \mathcal{V}_k, \quad \mathbf{x}(0) \in \mathcal{X}_0. \tag{2}$$

In this work, we assume that $\mathcal{W}_k, \mathcal{V}_k, \mathcal{X}_0$ are compact basic semi-algebraic sets described in terms of polynomial inequalities, namely:

$$\mathcal{W}_k = \{\mathbf{w}(k) \in \mathbb{R}^n : \; h_i^w(\mathbf{w}(k), k) \leq 0, \; i = 1, \ldots, t_w\}, \tag{3}$$

where h_i^w (with $i = 1, \ldots, t_w$, $t_w \in \mathbb{N}$) are known polynomial functions in the variable $\mathbf{w}(k)$. The sets $\mathcal{V}_k, \mathcal{X}_0$ are described similarly. Note that the assumption that the sets $\mathcal{W}_k, \mathcal{V}_k, \mathcal{X}_0$ are semialgebraic is quite general and includes, among others, uncertainty sets such as bounding boxes and ellipsoidal regions.

This paper addresses a set-values filtering problem, formally defined as follows.

Problem 1. **[Set-valued filtering]**
Given the dynamical equations (1), the output observation $\mathbf{y}(k)$, the bounding sets for the initial state and for the noises $\mathcal{X}_0, \mathcal{W}_k, \mathcal{V}_k$, the problem of set-valued filtering aims at recursively computing, for each time step $k = 1, 2, \ldots$, the state uncertainty set $\mathcal{X}_k$ defined as:

$$\mathcal{X}_k = \{\, \mathbf{x}(k) \in \mathbb{R}^n\!: \; \mathbf{x}(k) - \mathbf{a}(\mathbf{x}(k-1), k-1) \in \mathcal{W}_{k-1}, \; \mathbf{y}(k) - \mathbf{c}(\mathbf{x}(k), k) \in \mathcal{V}_k, \; \mathbf{x}(k-1) \in \mathcal{X}_{k-1} \,\}. \tag{4}$$

■

[1] Without loss of generality, autonomous systems not driven by external inputs are considered in (1). The presence of known input signals can be simply embedded in the time-varying coefficients of the polynomials $\mathbf{a}(\mathbf{x}(k), k)$ and $\mathbf{c}(\mathbf{x}(k), k)$.

Note that the state uncertainty set $\mathcal{X}_k$ in (4) is defined in a recursive way, and it represents the set of all values of the state $\mathbf{x}(k)$ compatible with the available information, namely the system equations (1), the output observations up to time k, and the assumed uncertainty sets on the initial state $\mathbf{x}(0)$ and on the noises $\mathbf{w}(k-1)$ and $\mathbf{v}(k)$.

Unfortunately, the representation of the set $\mathcal{X}_k$ becomes more and more complicated as the time k increases. In the following, we propose an algorithm to compute, in a recursive fashion, an outer approximation $\mathcal{S}_k$ for the state uncertainty set $\mathcal{X}_k$ which is guaranteed to contain the "true" state $\mathbf{x}(k)$, *i.e.*, $\mathcal{S}_k \supseteq \mathcal{X}_k \in \mathbf{x}(k)$. More specifically, we propose a heuristic to find a semialgebraic set $\mathcal{S}_k$, described by an a-priori fixed number of polynomial inequalities, which outer approximates $\mathcal{X}_k$.

3. RECURSIVE APPROXIMATION OF THE STATE UNCERTAINTY SET $\mathcal{X}_K$

In order to recursively approximate the state uncertainty set $\mathcal{X}_k$ with an outer-bounding semialgebraic set $\mathcal{S}_k$, the outer-approximating set $\mathcal{S}_{k-1} \supseteq \mathcal{X}_{k-1}$ computed at the previous time step $k-1$ is propagated. This allows us to embed all past information into the set $\mathcal{S}_{k-1}$. In this way, we can easily construct a first outer approximation $\bar{\mathcal{X}}_k$ of the set $\mathcal{X}_k$ (originally defined in (4)) as follows

$$\bar{\mathcal{X}}_k = \{\, \mathbf{x}(k) \in \mathbb{R}^n\!: \; \mathbf{x}(k) - \mathbf{a}(\mathbf{x}(k-1), k-1) \in \mathcal{W}_{k-1}, \; \mathbf{y}(k) - \mathbf{c}(\mathbf{x}(k), k) \in \mathcal{V}_k, \; \mathbf{x}(k-1) \in \mathcal{S}_{k-1} \,\}. \tag{5}$$

Note that, since $\mathcal{W}_{k-1}$ and $\mathcal{V}_k$ are assumed to be semialgebraic sets, and $\mathcal{S}_{k-1}$ is semialgbraic by construction, thus $\bar{\mathcal{X}}_k$ is a semialgebraic set, defined in a compact form by generic polynomial inequality constraints as follows

$$\bar{\mathcal{X}}_k = \{\mathbf{x}(k) \in \mathbb{R}^n : h_i^x(\tilde{\mathbf{x}}(k), k) \leq 0, \quad i = 1, \ldots, t_x\}, \tag{6}$$

where $h_i^x(\tilde{\mathbf{x}}(k), k)$, with $i = 1, \ldots, t_x$, are polynomial functions in the augmented state variable $\tilde{\mathbf{x}}(k) = [\mathbf{x}(k) \;\; \mathbf{x}(k-1)]$ and they are defined based on the representation of $\bar{\mathcal{X}}_k$ in (5).

4. OUTER APPROXIMATING $\bar{\mathcal{X}}_K$

In this section, we show how to outer approximate $\bar{\mathcal{X}}_k$ (and thus also the tight state uncertainty set $\mathcal{X}_k$) by a more simple semialgeraic set $\mathcal{S}_k$ defined by an a-priori specified number $t_s \in \mathbb{N}$ of polynomial inequalities of a given degree less than or equal to $d \in \mathbb{N}$.

Let us first consider the following generic description for the set $\mathcal{S}_k$ we are seeking for:

$$\mathcal{S}_k = \{\mathbf{x}(k) \in \mathbb{R}^n : \; h_i^s(\mathbf{x}(k), k) \leq 0, \quad i = 1, \ldots, t_s\}, \tag{7}$$

where $h_i^s(\mathbf{x}(k), k)$, with $i = 1, \ldots, t_s$, are polynomial functions in the state $\mathbf{x}(k)$ of maximum degree d, *i.e.*,

$$h_i^s(\mathbf{x}(k), k) = \mathbf{h}_i(k)\mathbf{q}_d(\mathbf{x}(k)), \tag{8}$$

where

$$\mathbf{q}_d(\mathbf{x}) = [1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d]^\top \tag{9}$$

is the vector (of dimension $s(d) = \binom{n+d}{d}$) containing all monomials of degrees less than or equal to d, and $\mathbf{h}_i(k) \in$

$\mathbb{R}^{1\times s(d)}$ is a row vector of coefficients. In the following, the dependence of $\mathcal{S}_k$, $\bar{\mathcal{X}}_k$, $h_i^s(\mathbf{x}(k),k)$ and $\mathbf{h}_i(k)$ on the time index k will be dropped to alleviate the notation, and used only when needed.

Our goal is thus to choose the polynomials $h_i^s(\mathbf{x})$ (or equivalently the corresponding vector of coefficients $\mathbf{h}_i$), with $i = 1,\ldots,t_s$, solving the following minimum-volume problem:

$$\inf_{\mathcal{S}} \int_{\mathcal{S}} d\mathbf{x} \quad \text{s.t.} \quad \bar{\mathcal{X}} \subseteq \mathcal{S}, \tag{10}$$

where $\mathcal{S}$ is constrained to be a semialgebraic set defined as in (7).

In the following, we present a sequential heuristic to compute an approximate solution of the minimum-volume problem (10). Although approximated, the computed solution always satisfies the hard constraint $\bar{\mathcal{X}} \subseteq \mathcal{S}$, and thus the state $\mathbf{x}(k)$ is guaranteed to belong to the computed set $\mathcal{S}_k$ for all time indexes $k = 1, 2, \ldots$.

4.1 Approximating the objective function in (10)

Computing the volume $\int_{\mathcal{S}} d\mathbf{x}$ of a semialgebraic set $\mathcal{S}$ is a challenging problem even in the simpler case where $\mathcal{S}$ is a polytope (see, e.g., (Dyer and Frieze, 1988; Bueler et al., 2000; Wiback et al., 2004) where algorithms to approximate the volume of a polytope are presented). In our case, the problem is even more difficult as $\mathcal{S}$ is not known, as computing $\mathcal{S}$ is part of the problem itself. The first step we perform in this paper to tackle this challenging problem is based on Monte Carlo integration, as described in the following.

Let us suppose we are given an outer-bounding box $\mathcal{B} \subseteq \mathbb{R}^n$ containing the set $\bar{\mathcal{X}}$ we would like to outer approximate (the computation of the box $\mathcal{B}$ will be discussed in Section 4.5). Then, a sequence $\mathcal{P} = \{p_j\}_{j=1}^N$ of random points independent and uniformly distributed in $\mathcal{B}$ is generated and, according to Monte Carlo integration, the volume of the (unknown) set $\mathcal{S}$ is approximated by

$$\int_{\mathcal{S}} d\mathbf{x} \approx \frac{1}{N} Vol(\mathcal{B}) \sum_{p_j \in \mathcal{P}} I_{\{\mathcal{S}\}}(p_j), \tag{11}$$

where $Vol(\mathcal{B})$ is the volume of the box $\mathcal{B}$ and $I_{\{\mathcal{S}\}}(p_j)$ is the indicator function of the set $\mathcal{S}$ defined as

$$I_{\{\mathcal{S}\}}(p_j) = \begin{cases} 1 & \text{if } p_j \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Remark 1. It is well known from the theory of Monte Carlo integration (see, e.g., (Robert and Casella, 2004)) that the following limit holds *with probability* 1:

$$\lim_{N\to\infty} \frac{1}{N} Vol(\mathcal{B}) \sum_{p_j \in \mathcal{P}} I_{\{\mathcal{S}\}}(p_j) = \int_{\mathcal{S}} d\mathbf{x}. \tag{13}$$

■

Using (11), the minimum-volume problem (10) is approximated by

$$\inf_{\mathcal{S}} \sum_{p_j \in \mathcal{P}} I_{\{\mathcal{S}\}}(p_j) \quad \text{s.t.} \quad \bar{\mathcal{X}} \subseteq \mathcal{S}. \tag{14}$$

The minimization problem (14) is solved through the sequential procedure described in the following paragraph.

Algorithm 1. Sequential procedure for outer approximating $\bar{\mathcal{X}}$

Input: Number t_s of inequality constraints $h_i^s(\mathbf{x}) \leq 0$ defining $\mathcal{S}$ in (7) ; sequence $\mathcal{P} = \{p_j\}_{j=1}^N$ of random points uniformly distributed in an outer-bounding box $\mathcal{B}$.

1. **for** $i = 1,\ldots,t_s$
   - 1.1. **compute** the coefficient vector $\mathbf{h}_i^s \neq 0$ of the polynomial $h_i^s(\mathbf{x}) \neq 0$ such that: (*i*) $h_i^s(\mathbf{x}) \leq 0 \ \forall \ \mathbf{x} \ \in \ \bar{\mathcal{X}}$; (*ii*) the number of points $p_j \in \mathcal{P}$ satisfying the constraint $h_i^s(p_j) \leq 0$ is minimized, *i.e.*,
     $$\mathbf{h}_i^s = \arg \min_{\mathbf{h}_i^s \neq 0} \sum_{p_j \in \mathcal{P}} I_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j) \tag{15a}$$
     $$s.t. \quad h_i^s(\mathbf{x}) \leq 0 \quad \forall \ \mathbf{x} \ \in \ \bar{\mathcal{X}} \tag{15b}$$
   - 1.2. **update** the sequence $\mathcal{P}$ by collecting all and only points in $\mathcal{P}$ which satisfy the constraint $h_i^s(p_j) \leq 0$
2. **end for**
3. **define** $\mathcal{S}$ as
   $$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : \ h_i^s(\mathbf{x}) \leq 0, \ \ i = 1,\ldots,t_s\} \tag{16}$$

Output: Outer approximating semialgebraic set $\mathcal{S}$.

4.2 A sequential procedure for solving (14)

The sequential procedure presented in this paragraph to compute a feasible semialgebraic set $\mathcal{S}$ for (14) (and with the fixed structure in (7)) is outlined in Algorithm 1.

Algorithm 1 generates, in a sequential way, a sequence of semialgebraic sets $S^{(i)}$, with $i = 1,\ldots,t_s$, each defined by a single polynomial inequality constraint as follows

$$\mathcal{S}^{(i)} = \{\mathbf{x}(k) \in \mathbb{R}^n : \ h_i^s(\mathbf{x}) \leq 0\}. \tag{17}$$

First, the set $\mathcal{S}^{(1)}$ which minimizes (an approximation of) the volume of the set $\mathcal{B} \cap S^{(1)}$ is computed (Step 1.1, Problem (15)). We remind that the approximation is due to the fact that $Vol(\mathcal{B} \cap S^{(1)})$ is approximated, up to the constant $\frac{1}{N}Vol(\mathcal{B})$, by $\sum_{p_j \in \mathcal{P}} I_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j)$ in the objective cost (15a). Because of the constraint (15b), the set $S^{(1)}$ is guaranteed to contain the original semialgebraic set $\bar{\mathcal{X}}$, *i.e.*, $\bar{\mathcal{X}} \subseteq S^{(1)}$.

Then, a new semialgebraic region $\mathcal{S}^{(2)} \supseteq \bar{\mathcal{X}}$ that minimizes (an approximation of) the volume of the set $\mathcal{B} \cap \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ is generated. In order to approximate the volume of $\mathcal{B} \cap \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, all points p_j in the sequence $\mathcal{P}$ that do not belong to $\mathcal{B} \cap \mathcal{S}^{(1)}$ were previously discarded (Step 1.2).

The procedure is repeated until t_s sets $\mathcal{S}^{(i)}$ are generated, with final outer-approximating set $\mathcal{S}$ defined by

$$\mathcal{S} = \cap_{i=1}^{t_s} \mathcal{S}^{(i)}. \tag{18}$$

Remark 2. The constraints defining the box $\mathcal{B}$ can be also added in the definition $\mathcal{S}$ to further reduce the volume of $\mathcal{S}$. In this case, the final outer approximating set $\mathcal{S}$ is given by

$$\mathcal{S} = \mathcal{B} \cap \left(\cap_{i=1}^{t_s} \mathcal{S}^{(i)} \right),$$

leading to a set $\mathcal{S}$ defined by $2n + t_s$ polynomial inequalities constraints ($2n$ of which are actually linear). ■

Remark 3. In order to handle the constraint $\mathbf{h}_i^s \neq 0$ and normalize the vector $\mathbf{h}_i^s$, we may split problem (15) into two problems, where we may enforce, for instance, that the coefficient associated to the monomial x_1 is 1 (resp. -1). Because of the simplicity in handling the constraint $\mathbf{h}_i^s \neq 0$, in the following, we some abuse of notation, we will omit the condition $\mathbf{h}_i^s \neq 0$. ■

The following paragraph provides all mathematical details to compute a solution of problem (15), the core step in Algorithm 1. In particular, we show how to handle the discontinuous objective function $\sum_{p_j \in \mathcal{P}} I_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j)$ and the robust constraint $h_i^s(\mathbf{x}) \leq 0 \quad \forall\, \mathbf{x} \in \bar{\mathcal{X}}$.

4.3 Approximation of the indicator functions

Let us first consider the discontinuous objective function in (15a), given by the sum of the indicator functions $I_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j)$ defined as

$$I_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j) = \begin{cases} 1 & \text{if } h_i^s(p_j) \leq 0, \\ 0 & \text{if } h_i^s(p_j) > 0. \end{cases} \tag{19}$$

Each indicator function is approximated by the continuous piecewise polynomial function $T_{\{h_i^s(\mathbf{x}) \leq 0\}}$ defined as

$$T_{\{h_i^s(\mathbf{x}) \leq 0\}}(\mathbf{x}) = \begin{cases} -h_i^s(\mathbf{x}) & \text{if } h_i^s(\mathbf{x}) \leq 0, \\ 0 & \text{if } h_i^s(\mathbf{x}) > 0. \end{cases} \tag{20}$$

Based on the approximations of the indicator functions $I_{\{h_i^s(\mathbf{x}) \leq 0\}}$ with $T_{\{h_i^s(\mathbf{x}) \leq 0\}}$, problem (15) is relaxed by the following problem

$$\mathbf{h}_i^s = \arg \min_{\mathbf{h}_i^s \neq 0} \sum_{p_j \in \mathcal{P}} T_{\{h_i^s(\mathbf{x}) \leq 0\}}(p_j) \tag{21a}$$

$$s.t. \quad h_i^s(\mathbf{x}) \leq 0 \quad \forall\, \mathbf{x} \in \bar{\mathcal{X}} \tag{21b}$$

Note that the functional of problem (21) is a convex piecewise linear function in the unknown optimization variables (namely, the coefficients $\mathbf{h}_i^s$ of the polynomial $h_i^s(\mathbf{x})$).

Theorem 1. If there exists at least one point p_j in the sequence $\mathcal{P}$ such that $h_i^s(p_j) < 0$ (with h_i^s being the optimal polynomial solving problem (21)), then the set $h_i^s(\mathbf{x}) = 0$ and $\bar{\mathcal{X}}$ have (at least) a common point.

Proof: Theorem 1 is proven by contradiction. Let $\tilde{h}_i^s(\mathbf{x})$ be the optimal polynomial function for (21) and suppose that all the roots of $\tilde{h}_i^s(\mathbf{x})$ do not belong to $\bar{\mathcal{X}}$, that is: $\tilde{h}_i^s(\mathbf{x}) < 0$ for all $\mathbf{x} \in \bar{\mathcal{X}}$. Equivalently, for some $\varepsilon > 0$,

$$\tilde{h}_i^s(\mathbf{x}) + \varepsilon \leq 0 \quad \forall \mathbf{x} \in \bar{\mathcal{X}}. \tag{22}$$

Let us define the polynomial $\tilde{\tilde{h}}_i^s$ as $\tilde{\tilde{h}}_i^s(\mathbf{x}) = \tilde{h}_i^s(\mathbf{x}) + \varepsilon$. From (22), the polynomial $\tilde{\tilde{h}}_i^s(\mathbf{x})$ is feasible for problem (21). Furthermore, since the constraint $\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0$ is more demanding than $\tilde{h}_i^s(\mathbf{x}) \leq 0$, we have $\tilde{\tilde{\mathcal{S}}}^{(i)} \subseteq \tilde{\mathcal{S}}^{(i)}$, with

$$\tilde{\mathcal{S}}^{(i)} = \left\{ \mathbf{x}(k) \in \mathbb{R}^n : \ \tilde{h}_i^s(\mathbf{x}) \leq 0 \right\},$$

$$\tilde{\tilde{\mathcal{S}}}^{(i)} = \left\{ \mathbf{x}(k) \in \mathbb{R}^n : \ \tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0 \right\}.$$

Let us define the costs $\tilde{V} = \sum_{p_j \in \mathcal{P}} T_{\{\tilde{h}_i^s(\mathbf{x}) \leq 0\}}(p_j)$ and $\tilde{\tilde{V}} = \sum_{p_j \in \mathcal{P}} T_{\{\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0\}}(p_j)$. By definition of the function $T_{\{h_i^s(\mathbf{x}) \leq 0\}}(\mathbf{x})$ in (20), we have

$$T_{\{\tilde{h}_i^s(\mathbf{x}) \leq 0\}}(\mathbf{x}) = \begin{cases} -\tilde{h}_i^s(\mathbf{x}) & \text{if } \mathbf{x} \in \tilde{\mathcal{S}}^{(i)}, \\ 0 & \text{if } \mathbf{x} \notin \tilde{\mathcal{S}}^{(i)}. \end{cases} \tag{23}$$

and

$$T_{\{\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0\}}(\mathbf{x}) = \begin{cases} -\tilde{\tilde{h}}_i^s(\mathbf{x}) & \text{if } \mathbf{x} \in \tilde{\tilde{\mathcal{S}}}^{(i)}, \\ 0 & \text{if } \mathbf{x} \notin \tilde{\tilde{\mathcal{S}}}^{(i)}. \end{cases} \tag{24}$$

Since $\tilde{\tilde{\mathcal{S}}}^{(i)} \subseteq \tilde{\mathcal{S}}^{(i)}$, then:

- when $T_{\{\tilde{h}_i^s(\mathbf{x}) \leq 0\}}(p_j) = 0$, also $T_{\{\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0\}}(p_j)$ is equal to 0.
- when $T_{\{\tilde{h}_i^s(\mathbf{x}) \leq 0\}}(p_j) = -\tilde{h}_i^s(p_j)$, then $T_{\{\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0\}}(p_j)$ can be equal either to 0 or to $-\tilde{\tilde{h}}_i^s(p_j)$. In this case, since $-\tilde{h}_i^s(p_j) \geq 0$, and $-\tilde{h}_i^s(p_j) > -\tilde{\tilde{h}}_i^s(p_j)$, then $T_{\{\tilde{h}_i^s(\mathbf{x}) \leq 0\}}(p_j) > T_{\{\tilde{\tilde{h}}_i^s(\mathbf{x}) \leq 0\}}(p_j)$.

By hypothesis, there should exist at least one point p_j in the sequence $\mathcal{P}$ such that $\tilde{h}_i^s(p_j) < 0$. Thus, based on the second condition above, we have $\tilde{V} > \tilde{\tilde{V}}$. Therefore, $\tilde{h}_i^s$ can not be the optimal solution of problem (21). This leads to a contradiction. ■

The statement in Theorem 1 can be interpreted as follows. Although the indicator functions have been relaxed by the convex-in-the-coefficients functions $T_{\{h_i^s(\mathbf{x}) \leq 0\}}$, the optimization problem (21) provides a semialgebraic set $\mathcal{S}^{(i)}$ which outer approximates $\bar{\mathcal{X}}$ and always passes through at least a point of the boundary of $\bar{\mathcal{X}}$.

4.4 Handling the constraint $\bar{\mathcal{X}} \subseteq \mathcal{S}^{(i)}$

The constraint $\bar{\mathcal{X}} \subseteq \mathcal{S}^{(i)}$, or equivalently the robust inequality constraint $h_i^s(\mathbf{x}) \leq 0 \quad \forall\, \mathbf{x} \in \bar{\mathcal{X}}$ in (21b), can be handled through relaxation techniques based on *sum-of-squares* (SOS) polynomials. Indeed, a sufficient condition for a polynomial to be nonnegative over a semialgebraic set is that it can be written in terms of SOS polynomials, as described in the following.

First, the definition of a SOS polynomial is provided.

Definition 1. A polynomial σ in the variable $\mathbf{x} \in \mathbb{R}^n$ of degree $2\mathbf{d}$ is a sum-of-square polynomial if and only if it can be written as:

$$\sigma(\mathbf{x}) = \mathbf{q}_\mathbf{d}(\mathbf{x})^\top \mathbf{Q} \mathbf{q}_\mathbf{d}(\mathbf{x}), \tag{25}$$

with $\mathbf{Q}$ being a real positive-semidefinite symmetric matrix (of size $\binom{n+\mathbf{d}}{\mathbf{d}}$), and $\mathbf{q}_\mathbf{d}(\mathbf{x})$ is the vector of all monomials up to degree $\mathbf{d}$ and defined similarly to (9). ■

Using relaxations based on sum-of-square polynomials (see, *e.g.*, (Parrilo, 2003)), Problem (21) is relaxed by the following problem:

$$
\begin{aligned}
\underline{\mathbf{h}}_i^s = \arg \min_{\substack{\mathbf{h}_i^s \\ \{\mathbf{Q}_t\}_{t=0}^{t_x}}} & \sum_{p_j \in \mathcal{P}} T_{\{h_i^s(\mathbf{x}) \le 0\}}(p_j) & \text{(26a)} \\
s.t. \quad & \mathbf{h}_i^s \mathbf{q}_d(\mathbf{x}) = -\mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_0 \mathbf{q_d}(\tilde{\mathbf{x}}) + \\
& + \sum_{t=1}^{t_x} \mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_t \mathbf{q_d}(\tilde{\mathbf{x}}) h_i^x(\tilde{\mathbf{x}}), \quad \forall\, \tilde{\mathbf{x}} \in \mathbb{R}^{2n} & \text{(26b)} \\
& \mathbf{Q}_t \succeq 0, \quad t = 0, \ldots, t_x. & \text{(26c)}
\end{aligned}
$$

Note that (26) is a convex *semidefinite programming* (SDP) problem. In fact: (i) the objective function is a convex piecewise-linear function in the unknown optimization variables $\mathbf{h}_i^s$; (ii) checking if the polynomial $h_i^s(\mathbf{x}) = \mathbf{h}_i^s \mathbf{q}_d(\mathbf{x})$ is equal to $-\mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_0 \mathbf{q_d}(\tilde{\mathbf{x}}) + \sum_{t=1}^{t_x} \mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_t \mathbf{q_d}(\tilde{\mathbf{x}}) h_i^x(\tilde{\mathbf{x}})$ for all $\tilde{\mathbf{x}}$ leads to linear equality constraints in the unknown coefficients $\mathbf{h}_i^s$ and in the matrix coefficients $\mathbf{Q}_t$ of the SOS polynomials $\sigma_t(\tilde{\mathbf{x}}) = \mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_t \mathbf{q_d}(\tilde{\mathbf{x}})$.

Furthermore, from the definition of the set $\bar{\mathcal{X}}$ in (7), the polynomials $h_i^x(\tilde{\mathbf{x}})$ are always negative for all $\tilde{\mathbf{x}} \in \bar{\mathcal{X}}$. Thus, the term $-\mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_0 \mathbf{q_d}(\tilde{\mathbf{x}}) + \sum_{t=1}^{t_x} \mathbf{q_d}(\tilde{\mathbf{x}})^\top \mathbf{Q}_t \mathbf{q_d}(\tilde{\mathbf{x}}) h_i^x(\tilde{\mathbf{x}})$ is always negative for all $\tilde{\mathbf{x}} \in \bar{\mathcal{X}}$. Because of the equality constraint in (26b), the original robust constraint $h_i^s(\mathbf{x}) \le 0 \quad \forall\, \mathbf{x} \in \bar{\mathcal{X}}$ in (21b) is always satisfied. Thus, the solution $\underline{\mathbf{h}}_i^s$ of the SDP problem (26) is always a feasible solution for the semi-infinite optimization problem (21).

Since the equality constraint in (26b) gives only a sufficient condition for the negativity of $h_i^s(\mathbf{x})$ on $\bar{\mathcal{X}}$, it follows that conservativeness is introduced in solving (26) instead of (21). Thus, the relaxed semialgebraic set $\underline{\mathcal{S}}^{(i)}$ defined as

$$
\underline{\mathcal{S}}^{(i)} = \{\mathbf{x}(k) \in \mathbb{R}^n : \ \underline{\mathbf{h}}_i^s \mathbf{q}_d(\mathbf{x}) \le 0\},
$$

is guaranteed to contain $\bar{\mathcal{X}}$, *i.e.*, $\bar{\mathcal{X}} \subseteq \underline{\mathcal{S}}^{(i)}$.

Finally, according to the *Putinar's Positivstellensatz* (see, e.g., (Putinar, 1993)), a polynomial which is nonnegative over a compact semialgebraic set $\bar{\mathcal{X}}$ can be written (under mild assumptions) as a combination of SOS polynomials, provided that the degree $2\mathbf{d}$ of the SOS polynomials $\sigma_0(\tilde{\mathbf{x}}), \ldots, \sigma_{t_x}(\tilde{\mathbf{x}})$ is large enough. Thus, the conservatives of the SDP problem (26) in approximating (21) can be reduced by increasing $\mathbf{d}$. Nevertheless, in practice, satisfactory results are achieved for small values of the SOS degree $2\mathbf{d}$.

Remark 4. Thanks to the flexibility of semialgebraic sets, ellipsoidal outer approximating regions can be also computed. This simply requires to use only one second-order inequality constraint $\underline{\mathbf{h}}_1^s \mathbf{q}_d(\mathbf{x}) \le 0$, and enforce the Hessian of $\underline{\mathbf{h}}_1^s \mathbf{q}_d(\mathbf{x})$ to be positive semidefinite. This leads to an addition convex *linear matrix-inequality* (LMI) in the unknown coefficients $\underline{\mathbf{h}}_1^s$. ■

4.5 Computing the initial outer-bounding box $\mathcal{B}$

So far, we have assumed to know an outer-bounding box $\mathcal{B}$ for the set $\bar{\mathcal{X}}$ from which the initial sequence $\mathcal{P}$ of random points is generated. The computation of this box $\mathcal{B}$ can be easily formulated in terms of $2n$ independent polynomial optimization problems (namely, one minimization and one optimization problem for each dimension of $\bar{\mathcal{X}}$), as also recently proposed in the literature of parametric set-membership identification (Cerone et al., 2013a,b). For instance, the maximum value $\overline{x}_i$ over the i-th axis, namely

$$
\overline{x}_i = \max_{x \in \bar{\mathcal{X}}} x_i, \tag{27a}
$$

is given by the solution of the semi-infinite optimization problem

$$
\begin{aligned}
& \min_{\rho} \rho, & \text{(28a)} \\
& s.t. \quad x_i - \rho \le 0 \quad \forall\, x \in \ \bar{\mathcal{X}}. & \text{(28b)}
\end{aligned}
$$

An upper bound of $\overline{x}_i$ can be thus obtained by replacing the robust constrain $x_i - \rho \le 0 \quad \forall\, x \in \ \bar{\mathcal{X}}$ using the SOS relaxation discussed in Section 4.4.

5. NUMERICAL EXAMPLE

Let us consider the Van der Pol oscillator, whose dynamics are described in continuous time by the nonlinear differential equations:

$$
\begin{aligned}
\dot{x}_1 &= x_2, & \text{(29a)} \\
\dot{x}_2 &= \mu x_2(1 - x_1^2) - x_1, & \text{(29b)}
\end{aligned}
$$

where x_1, x_2 are the two state variables of the system, and μ is the damping parameter, which is set equal to 0.5 in this example. The system (29) is simulated for 6 seconds starting from initial conditions $x_1(0) = 0.5$ and $x_2(0) = -0.5$. Only the first state variable x_1 is assumed to be measured at a sampling time $T_s = 0.2$ s. Thus, for each sampling time $k = 1, 2, \ldots$, the output equation of the system is given by

$$
\mathbf{y}(k) = x_1(k) + \mathbf{v}(k), \tag{30}
$$

where the amplitude of the measurement noise $\mathbf{v}$ is assumed to be bounded by $|\mathbf{v}(k)| \le 0.2$.

The discrete-time representation of the model is obtained by approximating the differential equations (29) using a forward Euler method. No process noise is assumed (*i.e.*, $\mathbf{w}(k) = 0$) and the initial conditions are supposed to belong to the box $\mathcal{X}_0 = [0.2 \ 0.8] \times [-0.6 \ -0.2]$.

Semialgebraic outer approximations $\mathcal{S}_k$ of the state uncertainty set $\bar{\mathcal{X}}_k$ are computed for each time stamp $k = 1, 2, \ldots, 30$ via Algorithm 1, using a sequence $\mathcal{P}$ of $N = 20$ randomly generated points. Only $t_s = 2$ inequality constraints (plus 4 linear constraints defining the outer approximating box $\mathcal{B}_k$) are used to describe the set $\mathcal{S}_k$. Thus, in total, six SDP relaxed problems (26c) are solved. Such relaxed problems are formulated using the Matlab interface *SOStools* (Papachristodoulou et al., 2013), and solved through the general-purpose SDP solver *SeDuMi*. The average CPU time required to compute an outer-approximating semialgebraic set $\mathcal{S}_k$ is 42 seconds in an i7 2.5-GHz Intel core processor with 16 GB of RAM running MATLAB R2018a. This time also includes the formulation of the relaxed problems (26c) through the *SOStools* interface. Note that, only 4 seconds out of 42 are necessary to solve the six semi-definite programming problems.

Figure 1 shows the true time-trajectory of the two state variables, along with the bounds obtained by propagating the semialgebraic sets $\mathcal{S}_k$ computed using the approach described in the paper. For comparison, the bounds on each state obtained by propagating boxes are also plotted. The obtained results show that, as expected, propagating

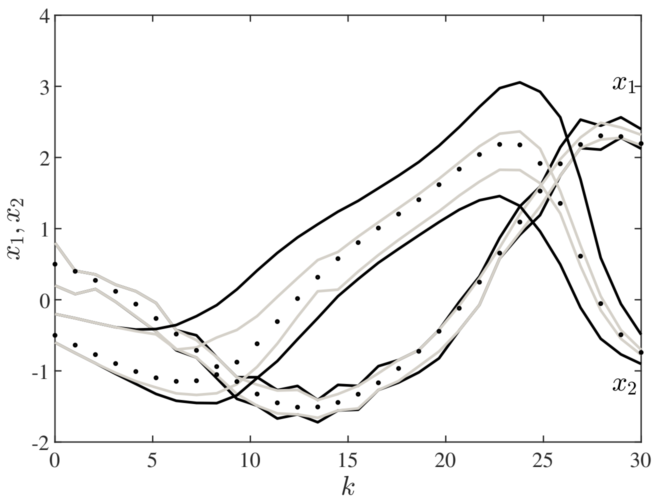

Fig. 1. Bounds on the state trajectories obtained by propagating semialgebraic sets (grey line) and boxes (black line); true state (black dots).

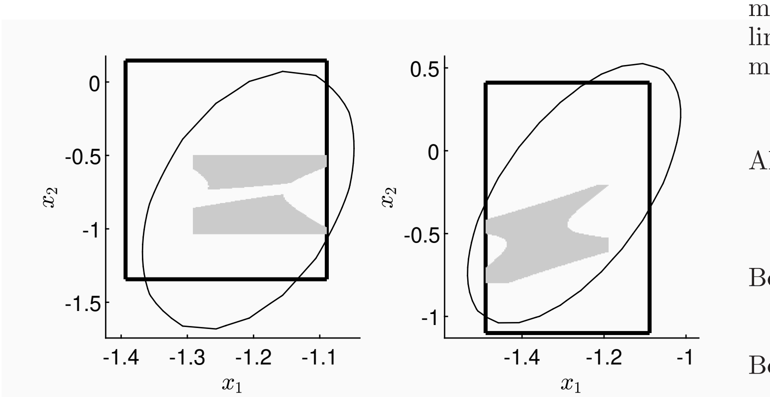

Fig. 2. Outer-approximations membership sets obtained by propagating: semialgebraic sets (gray region); boxes (thick line); ellipsoids (thin line). Left panel: set obtained at time $k = 9$. Right panel: set obtained at time $k = 10$.

outer-approximating semialgebraic sets instead of outer-approximating boxes provides a more accurate reconstruction of the state variables.

Figure 2 shows the outer-approximating semialgebraic sets $\mathcal{S}_k$ computed at time samples $k = 9$ and $k = 10$. For the sake of comparison, the outer-approximating sets obtained by propagating boxes and ellipsoidal regions (see Remark 4) are also reported. As point out in the Introduction, our method allows the practitioner to choose any shape (that can be described by semi-algebraic inequalities) as membership set. The boxes and ellipsoidal regions in Figure 2 are the result of the application of our method using such shapes. It is interesting to note that the computed semialgebraic membership sets $\mathcal{S}_k$ might not be connected and, although only $t_s = 2$ polynomial inequality constraints are considered, the obtained sets are described by more than two curved edges (not including the four axis-aligned edges defining the box $\mathcal{B}_k$). This is very important in nonlinear set-membership because the true membership set may be neither convex nor connected.

6. CONCLUSION

We have derived a flexible, accurate and efficient set-valued filtering algorithm for time-varying dynamical systems characterised by polynomial nonlinearities. Our approach allows the practitioner to select any semi-algebraic set as membership set. The parameters defining the inequalities describing the semi-algebraic set are selected by means of a sequential procedure that minimises a numerical estimate of the volume of the outer-bounding set. The key idea is to use a sum-of-square approximation, at each time instant, to formulate such minimization as a semi-definite programming problem. This means that the complexity of our algorithm is polynomial. Since bounding boxes, ellipsoidal regions, parallelotopes and zonotopes are semi-algebraic sets, previous approaches to set-membership can therefore be seen as particular cases of our methodology. Nevertheless, we stress the fact that our method can be applied to *nonlinear* dynamical systems characterised by polynomial nonlinearities.

As future work we plan to extend this approach to parametric estimation and to nonlinear systems whose nonlinearities can be described by piecewise polynomials or, more generally, semi-algebraic functions.

REFERENCES

Alamo, T., Bravo, J., and Camacho, E. (2003). Guaranteed state estimation by zonotopes. In *Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, USA*, 5831 – 5836.

Benavoli, A. (2013). The generalized moment-based filter. *IEEE Transactions on Automatic Control*, 58(10), 2642–2647.

Benavoli, A., Zaffalon, M., and Miranda, E. (2009). Reliable hidden markov model filtering through coherent lower previsions. In *Proc. 12th Int. Conf. Information Fusion*, 1743–1750. Seattle (USA).

Benavoli, A., Zaffalon, M., and Miranda, E. (2011). Robust filtering through coherent lower previsions. *IEEE Transactions on Automatic Control*, 56(7), 1567–1581.

Benavoli, A. and Piga, D. (2016). A probabilistic interpretation of set-membership filtering: Application to polynomial systems through polytopic bounding. *Automatica*, 70, 158–172.

Bertsekas, D. and Rhodes, I. (1971). Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, 16(2), 117 – 128.

Broman, V. and Shensa, M. (1990). A compact algorithm for the intersection and approximation of N-dimensional polytopes. *Mathematics and computers in simulation*, 32(5-6), 469–480.

Bueler, B., Enge, A., and Fukuda, K. (2000). Exact volume computation for polytopes: a practical study. In *DMV SEMINAR*, volume 29, 131–154. Springer.

Calafiore, G. (2005). Reliable localization using set-valued nonlinear filters. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, 35(2), 189–197.

Casini, M., Garulli, A., and Vicino, A. (2014). Feasible parameter set approximation for linear models with bounded uncertain regressors. *IEEE Transactions on Automatic Control*, 50(11), 2910–2920.

Cerone, V., Lasserre, J.B., Piga, D., and Regruto, D. (2014). A Unified Framework for Solving a General Class of Conditional and Robust Set-Membership Estimation Problems. *IEEE Transactions on Automatic Control*, 59(11), 2897–2909.

Cerone, V., Piga, D., and Regruto, D. (2013a). Computational load reduction in bounded error identification of Hammerstein systems. *IEEE Transactions on Automatic Control*, 58(5), 1317–1322.

Cerone, V., Piga, D., and Regruto, D. (2013b). A convex relaxation approach to set-membership identification of LPV systems. *Automatica*, 49(9), 2853 – 2859.

Chisci, L., Garulli, A., and Zappa, G. (1996). Recursive state bounding by parallelotopes. *Automatica*, 32:7, 1049–1055.

Chisci, L., Garulli, A., Vicino, A., and Zappa, G. (1998). Block recursive parallelotopic bounding in set membership identification. *Automatica*, 34(1), 15–22.

Combastel, C. (2003). A state bounding observer based on zonotopes. In *European Control Conference*.

Combettes, P.L. (1993). The foundations of set theoretic estimation. *Proceedings of the IEEE*, 81(2), 182–208.

Dabbene, F., Henrion, D., Lagoa, C., and Shcherbakov, P. (2015). Randomized approximations of the image set of nonlinear mappings with applications to filtering. In *IFAC Symposium on Robust Control Design (ROCOND 2015), Bratislava (Russia)*.

Deller, J.R., Nayeri, M., and Liu, M.S. (1994). Unifying the landmark developments in optimal bounding ellipsoid identification. *International Journal of Adaptive Control and Signal Processing*, 8(1), 43–60.

Dyer, M. and Frieze, A. (1988). On the complexity of computing the volume of a polyhedron. *SIAM Journal on Computing*, 17, 967–974.

El Ghaoui, L. and Calafiore, G. (2001). Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. *IEEE Transactions on Automatic Control*, 46(7), 1084–1089.

Kuntsevich, V. and Lychak, M. (1992). *Guaranteed estimates, adaptation and robustness in control systems*. Springer-Verlag.

Le, V.T.H., Alamo, T., Camacho, E.F., Stoica, C., and Dumur, D. (2011). A new approach for guaranteed state estimation by zonotopes. In *Proceedings of the 18th IFAC World Congress, Milano, Italy*, volume 28.

Maier, C. and Allgöwer, F. (2009). A set-valued filter for discrete time polynomial systems using sum of squares programming. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China*, 223–228.

Milanese, M. and Novara, C. (2011). Unified set membership theory for identification, prediction and filtering of nonlinear systems. *Automatica*, 47(10), 2141–2151.

Milanese, M. and Vicino, A. (1991). Optimal estimation theory for dynamic sistems with set membership uncertainty: an overview. *Automatica*, 27(6), 997–1009.

Mo, S. and Norton, J. (1990). Fast and robust algorithm to compute exact polytope parameter bounds. *Mathematics and computers in simulation*, 32(5-6), 481–493.

Papachristodoulou, A., Anderson, J., Valmorbida, G., Prajna, S., Seiler, P., and Parrilo, P.A. (2013). *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*. Url: http://www.eng.ox.ac.uk/control/sostools.

Parrilo, P. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96, 293–320.

Piet-Lahanier, H. and Walter, E. (1989). Further results on recursive polyhedral description of parameter uncertainty in the bounded-error context. In *Proceedings of the 28th IEEE Conference on Decision and Control, Tampa, Florida, USA*, 1964 –1966.

Puig, V., Saludes, J., and Quevedo, J. (2003). Worst-case simulation of discrete linear time-invariant interval dynamic systems. *Reliable Computing*, 9(4), 251–290.

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42, 969–984.

Robert, C. and Casella, G. (2004). *Monte Carlo statistical methods*. Springer Science.

Savkin, A. and Petersen, I. (1998). Robust state estimation and model validation for discrete-time uncertain systems with a deterministic description of noise and uncertainty. *Automatica*, 34(2), 271–274.

Schweppe, F.C. (1967). Recursive state estimation: Unknown but bounded errors and system inputs. In *Adaptive Processes, Sixth Symposium on*, volume 6, 102 –107.

Wiback, S., Famili, I., Greenberg, H., and Palsson, B. (2004). Monte Carlo sampling can be used to determine the size and shape of the steady-state flux space. *Journal of theoretical biology*, 228(4), 437–447.