

# Nonlinear Desirability as a Linear Classification Problem

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## Abstract

The present paper proposes a generalization of linearity axioms of coherence through a geometrical approach, which leads to an alternative interpretation of desirability as a *classification problem*. In particular, we analyze different sets of rationality axioms and, for each one of them, we show that proving that a subject, who provides finite accept and reject statements, respects these axioms, corresponds to solving a binary classification task using, each time, a different (usually nonlinear) family of classifiers. Moreover, by borrowing ideas from machine learning, we show the possibility to define a *feature mapping* allowing us to reformulate the above nonlinear classification problems as linear ones in a higher-dimensional space. This allows us to interpret gambles directly as payoff vectors of *monetary lotteries*, as well as to reduce the task of proving the rationality of a subject to a linear classification task.

**Keywords:** imprecise probabilities, coherence, convex coherence, monetary scale, piecewise separators

## 1. Introduction

The Bayesian framework is a sound and consistent theory because it is a *logic*. In fact, it can be shown that the rules of (Bayesian) probabilities can be inferred via mathematical duality from a set of *logical axioms* [3, 12, 23, 25, 30, 31], that one can interpret as rationality requirements in the way a subject accepts gambles on the results of an uncertain experiment. Mathematically, for a finite possibility space, gambles  $g$  are represented by column vectors in  $\mathbb{R}^n$ , where  $n$  is the size of the (finite) possibility space  $\Omega$  ( $n = |\Omega|$ ) of the experiment. In this paper, we interpret gambles as payoff vectors of *monetary lotteries*.<sup>1</sup> By denoting with  $\mathcal{D}$  the set of gambles an agent finds to be desirable and with  $g$  a generic gamble

in  $\mathbb{R}^n$ , the axioms of desirability can be expressed as:<sup>2</sup>

- D1: Tautologies if  $g \geq 0 \Rightarrow g \in \mathcal{D}$ ,
- D2: Falsum if  $g < 0 \Rightarrow g \notin \mathcal{D}$ ,
- D3: Linearity if  $g, h \in \mathcal{D} \Rightarrow \lambda g + \mu h \in \mathcal{D}$   
for any  $\lambda, \mu \in \mathbb{R}^+$ ,
- D4: Closure if  $g + \varepsilon \in \mathcal{D}$  for all  $\varepsilon \in \mathbb{R}_*^+ \Rightarrow g \in \mathcal{D}$ ,
- D5: Completeness if  $g \notin \mathcal{D} \Rightarrow -g \in \mathcal{D}$ .

The first axiom – D1 – also known as *accepting partial gains* criterion in literature [30], states that an agent should always accept non-negative gambles, because they can only increase the agent’s utility. The second axiom – D2 – also known as *avoiding sure loss* criterion in literature [30], states that an agent should always avoid negative gambles, because they can only decrease the agent’s utility. In what follows, for simplicity, we denote by  $T$  the set of all non-negative gambles  $T := \{g \in \mathbb{R}^n : g \geq 0\}$  and by  $F$  the set of negative gambles  $F := \{g \in \mathbb{R}^n : g < 0\}$ . Hence, the two previous axioms can be rewritten respectively as D1:  $T \subseteq \mathcal{D}$  and D2:  $\mathcal{D} \cap F = \emptyset$ . The third and fourth axiom state instead that the utility of the agent is linear – D3 (a standard assumption in decision theory) – and, respectively continuous: D4. D5 states that an agent has complete preferences on the states of the world.

These axioms provide the foundation of rational decision making as it was shown in [18, 32, 33], providing a connection between desirability and Von Neumann and Morgenstern’s [19] and Anscombe and Aumann’s [3] axiomatisation of rationality. However, in several situations the above axioms can be restrictive and researchers have relaxed and generalised them in many ways. For instance, it is not very realistic to assume that an agent can always compare alternatives. Axiomatisations of rational decision making under *incompleteness* can be obtained by dropping D5 [4, 13, 14, 18, 29, 30]. The closure (continuity) axiom D4 can also be abandoned. Although D4 makes tighter the connection between desirability and probability theory [5, 8, 10], the extra generality of sets of desirable gambles without D4 makes them useful when dealing with the problem of conditioning on sets of probability zero, or

1. This is not the usual interpretation. Walley [30] has discussed in some detail why gambles should be interpreted as uncertain rewards in terms of lottery tickets that can be won or lost. In this paper, we abandon this assumption because we aim to relax the linearity axiom.

2. In D4, we use  $\varepsilon$  to denote also the constant gamble given by  $g(\omega) = \varepsilon$  for every  $\omega \in \Omega$ .

of choosing between two options under zero expectation [28, 30]. The axioms D1 and D2 can also be restrictive. For instance, in case  $\Omega$  is an infinite possibility space, evaluating the positivity (or negativity) of a gamble can be computationally very demanding. This leads to a notion of computational rationality [6, 7] which restricts D1 and D2. Instead [26] focuses on relaxations of the notion of avoiding sure loss (D2) only. Finally, the axiom D3 (linearity) is also not very realistic, especially when one considers monetary gambles (i.e. gambles that return euros instead of lottery tickets about euros). The linearity axiom has been replaced by convexity in [17, 20].

In this paper, we assume that D1, D2 and D4 hold true and provide a *geometric* view of desirability which allows us to easily extend desirability to nonlinear utility.

In order to introduce this interpretation, we first consider a slightly more general framework for desirability whereby an agent can express both acceptance or rejection of gambles [22]. By rejecting a gamble, the agent expresses that they consider accepting that gamble unreasonable. Let  $\mathcal{A}$  and  $\mathcal{R}$  denote the set of acceptable and, respectively, rejectable gambles by the agent. We assume then the presence of a *modeller*, who aims to prove the *rationality* of the agent. The modeller can have in mind different concepts of rationality depending on the axioms they decide to consider as *rational*. Once established the axioms, they should evaluate the rationality of the agent on the basis of the only information available, which we assume to be  $(\mathcal{A}, \mathcal{R})$ . Clearly, in practical situations,  $\mathcal{A}$  and  $\mathcal{R}$  are finite. So, we assume that they consider the agent rational if there exists a set  $\mathcal{D}$  such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$  that satisfies the rationality axioms established by the modeller. In this paper, we assume that  $\mathcal{D}$  must always satisfy D1, D2 and D4 (which we consider minimal requirements for rationality), therefore  $T \subseteq \mathcal{D}$ ,  $\mathcal{D} \cap F = \emptyset$  and  $\mathcal{D}$  must be *closed* in the sense of D4. We will present rationality models where the modeller considers additional requirements of rationality and discuss methods to choose *the least committal* set  $\mathcal{D}$  compatible with these requirements –  $\mathcal{D}$  will be defined as *deductive closure* of  $\mathcal{A}$  taken with respect to the set of rationality axioms considered.<sup>3</sup>

It is well-known that, if the modeller assumes all axioms of rationality D1–D5, then the problem of determining if the agent is rational or not becomes, thanks to the hyperplane separation theorem, a binary linear classification problem [30]. Indeed, in this case the agent is rational if and only if there exists a *linear prevision*  $P(\cdot)$  such that  $P(g) \geq 0$ , for every  $g \in \mathcal{A}$  and  $P(g) < 0$ , for every  $g \in \mathcal{R}$ . However, a linear prevision is essentially an expected value operator taken with respect to a finitely additive probability:  $P(g) = g^T \beta$  for every  $g \in \mathbb{R}^n$  with  $\beta \in \mathbb{R}^n$ ,  $\beta \succeq 0$  and  $\sum_{i=1}^n \beta_i = 1$ , where  $\beta_i$ , for every  $i$ , are the components of the vector  $\beta$ . This is equivalent to have a binary linear clas-

sifier that classifies any gamble  $g$  on the basis of the sign of  $P(g) = g^T \beta$ , which classifies in a different way  $\mathcal{A}$  and  $\mathcal{R}$ . It is easy to see that the constraints on the coefficient  $\beta$  of this classifier, can be dropped asking it to directly classify in a different way  $\mathcal{A} \cup T$  and  $\mathcal{R} \cup F$ .<sup>4</sup> In this case, we say that the pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is *linearly separable*. Figure 1.1 provides a 2D illustration of this case.

In this paper, we show that this way of interpreting the problem as a binary classification task, is very general. In particular, indeed, we show that:

- in the *imprecise* case (that is, assuming D1–D4 but not *completeness* axiom D5), the problem can be reformulated using a *binary piecewise linear classifier* (see Section 2);
- if rationality is expressed through D1, D2, convexity replacing linearity, and D4 [17], the problem can be reformulated using a *binary piecewise (convex) affine classifier* (see Section 3);
- if rationality is expressed through D1, D2, “positive additivity” replacing linearity, and D4, the problem can be reformulated using a *binary piecewise positive affine classifier* (see Section 4);
- for more general cases that, however, respect D1, D2 and D4, the problem can be reformulate using a *non-linear classifier* (see Section 6).

This allows us to model more realistic cases of desirability, see Example 2. Moreover, by borrowing ideas from machine learning,<sup>5</sup> we show that we can define a *feature mapping* that allows us to reformulate the above nonlinear classification problems as linear classifiers in a higher-dimensional space.

## 2. Standard Imprecise Probability

Suppose to have a modeller who considers as *rational* an agent who respects D1-D4. Furthermore suppose, as before, that the only information they possess consist of a finite set of gambles  $\mathcal{A}$ , the agent is willing to accept, and a finite set of gambles  $\mathcal{R}$ , they are willing to reject. Then we can assume that the modeller considers the agent rational if and only if there exists a set of gambles  $\mathcal{D}$  that satisfies D1-D4 (that we call *coherent* from now on), such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ .

Let us indicate with  $\mathcal{L}(\Omega)$ , or  $\mathcal{L}$  when there is no possible ambiguity, the set of all gambles defined on a possibility space  $\Omega$ . Suppose moreover, as before, that  $|\Omega| = n$ . So, in what follows,  $\mathcal{L} = \mathbb{R}^n$ .

4. Indeed, if  $g^T \beta \geq 0$  for every  $g \in T$  and  $g^T \beta < 0$  for every  $g \in F$ , then  $\beta \succeq 0$ . Therefore, it can be normalized obtaining a probability distribution on  $\Omega$ .

5. There is however a substantial difference between the classification problem we aim to solving here and standard classification problems considered in machine learning. In our case, together with the training data set  $(\mathcal{A}, \mathcal{R})$ , there are two infinite sets  $T, F$  which must also be classified. To our best knowledge, this problem has only been studied by Mangasarian and Wild [16] for support vector machines.

3. We assume it is determined by  $\mathcal{A}$  and only indirectly by  $\mathcal{R}$ .

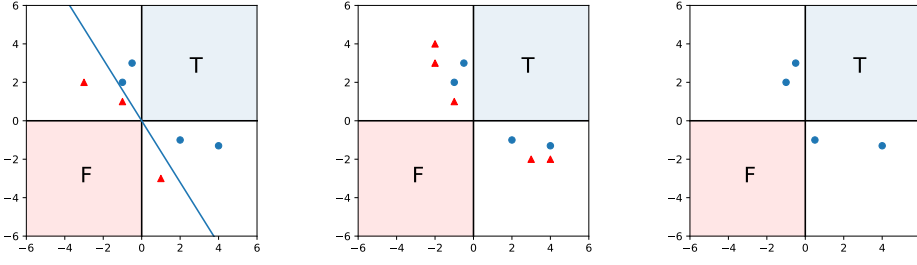


Figure 1.1: Example of a linearly separable pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , (left). The middle and right plots instead, show two examples of *non-linearly separable* pairs  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$ . The (blue) points denote gambles in  $\mathcal{A}$ ; the (red) triangles denote gambles in  $\mathcal{R}$ . The blue line in the left plot is the linear function  $g^T \beta = 0$  that separates  $\mathcal{A} \cup T$  from  $\mathcal{R} \cup F$ . In the right plot,  $\mathcal{R} = \emptyset$ .

In this section we prove that, determining whether an agent is rational in the sense of respecting D1–D4, is equivalent to solve a *binary piecewise linear* classification task.

**Definition 1 (Binary piecewise linear classifier)** We denote with the term *binary piecewise linear classifier*, a classifier  $PLC(\cdot)$  defined on  $\mathcal{L}$  and characterized by the following discriminant function:<sup>6</sup>

$$PLC(g) := \begin{cases} 1 & \text{if } g^T \beta_j \geq 0, \text{ for all } j \in \{1, \dots, N\} \\ -1 & \text{otherwise} \end{cases} \quad (1)$$

for every  $g \in \mathcal{L}$ , with  $\beta_j \in \mathbb{R}^n$ , for all  $j$ ,<sup>7</sup>  $N \geq 1$ .

**Definition 2 (Piecewise linear separability)** A pair of sets of gambles  $(A, B)$  is *piecewise linearly separable* if there exists a binary piecewise linear classifier  $PLC(\cdot)$ , such that  $PLC(A) = 1$  and  $PLC(B) = -1$ .<sup>8</sup> In this case, we indicate the set of these classifiers with  $PLC(A, B)$ .

Now we can show the main result of this section. All the proofs are in the supplementary material.

**Proposition 3** Given a pair of finite sets of gambles  $(\mathcal{A}, \mathcal{R})$ , there exists a coherent set  $\mathcal{D}$ , such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ , if and only if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise linearly separable.

From the proof of this proposition, it follows in particular that, if the pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise linearly separable, the region classified as 1 by a classifier  $PLC \in PLC(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , i.e.  $\mathcal{D} := \{g \in \mathcal{L} : PLC(g) = 1\}$ , is a coherent set, such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . Vice

6. We take for simplicity the values 1 and  $-1$ .

7. W.l.o.g. we can assume  $\beta_j \neq 0$  for every  $j$  since, otherwise, we can exclude it from the family of weights characterizing the classifier. This reasoning can be applied to every other classifier in the text.

8. With a little abuse of notation, with  $PLC(\mathcal{H}) = c$  for  $\mathcal{H} \subseteq \mathcal{L}$ ,  $c \in \{-1, 1\}$ , we mean  $PLC(g) = c$ , for all  $g \in \mathcal{H}$ . We will use the same notation also for the other types of binary classifiers considered later on.

versa is not true in general.<sup>9</sup> Analogous reasonings can be applied for the other concepts of rationality, and hence of *coherence*, treated in Section 3 and Section 4. Obviously, the same is valid also for the most strictly concept of rationality (expressed through axioms D1-D5), seen in the Introduction (it is a subcase of the actual one).

In this case however, we can give another interpretation of a rational agent, which is an extension of the one seen in the Introduction. It is easy to show indeed that every binary piecewise linear classifier that classifies  $T$  as 1, is characterized by a set of weights  $\{\beta_j\}_{j=1}^N$  such that  $\beta_j \geq 0$ , for all  $j = 1, \dots, N$ . Therefore, we can normalise them getting  $N$  probability mass functions on  $\Omega$ . So, given a pair of finite sets of gambles  $(\mathcal{A}, \mathcal{R})$ ,  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise linearly separable if and only if there exists a set of probability mass functions  $P_j = \beta_j$  on  $\Omega$  such that:  $E_{P_j}(g)$ , i.e. the expected value of a gamble  $g$  taken with respect to  $P_j$ , is not negative for all  $g \in \mathcal{A} \cup T$  and all  $P_j$ ; for all  $g \in \mathcal{R} \cup F$  there exists at least a  $P_i$  such that  $E_{P_i}(g) < 0$ . Reformulation of Prop. 3 in those probabilistic terms is also a well-known result in the literature [30].

Once established that the agent is rational, i.e. if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise linearly separable, the minimal set of assumptions the modeller can make on the agent's beliefs is represented by the least committal coherent set containing  $\mathcal{A}$  (see the proof of Prop. 3 or [30]):

$$\overline{\mathcal{E}(\mathcal{A})} := \overline{\text{posi}(\mathcal{A} \cup T)}, \quad (2)$$

where, given  $\mathcal{H} \subseteq \mathcal{L}$ :

$$\text{posi}(\mathcal{H}) := \left\{ \sum_{j=1}^r \lambda_j f_j : f_j \in \mathcal{H}, \lambda_j > 0, r \geq 1 \right\}, \quad (3)$$

and where  $\overline{\mathcal{H}'}$  of a set  $\mathcal{H}' \subseteq \mathcal{L}$  represents the closure of  $\mathcal{H}'$  with respect to the supremum norm topology [30, Section 3.7.2–3.7.4] or with respect to the usual topology of  $\mathbb{R}^n$  [30, Appendix D]. Notice that, geometrically, it corresponds to a (closed) convex cone. Further, it corresponds

9. For example if  $\mathcal{D}$  is not *finitely generated* [30].

also to a concept of *deductive closure* for the set  $\mathcal{A}$ . It is easy to see in fact that the operator  $C_{\text{cohr}} : \mathcal{A} \rightarrow \mathcal{E}(\mathcal{A})$ , defined for every  $\mathcal{A} \subseteq \mathcal{L}$ , is a closure operator. The proof of Prop. 3 guarantees in particular that this set can be rewritten as the region classified as 1 by a binary piecewise linear classifier. Hence, in principle, it can be found solving the classification task with some other constraints, which can be formulated as a linear programming problem [9].

## 2.1. Feature Mapping

In this section we will prove that the previous problem, which is a non linear classification task in general, can be reformulated as a linear classification problem in a higher dimensional space. Let  $\{\mathcal{B}_j\}_{j=1}^N$  denote a partition<sup>10</sup> of  $\mathcal{L}$  defined as follows [21]:

$$\mathcal{B}_j := \{g \in \mathcal{L} : g^T \omega_j \leq g^T \omega_k \text{ for } k = 1, \dots, N, j \neq k\}. \quad (4)$$

The vectors  $\omega_j \in \mathbb{R}^n$  are parameters defining the partition. Now, we can introduce the feature mapping  $\phi := (\phi_1, \dots, \phi_N)$ , with  $\phi_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined as  $\phi_j : g \rightarrow \mathbb{I}_{\mathcal{B}_j}(g)g$ , for every  $j = 1, \dots, N$ , where  $\mathbb{I}_{\mathcal{B}_j}(g) := 1$  if  $g \in \mathcal{B}_j$  and 0 otherwise.<sup>11</sup> Further, we define the following classifier, which corresponds to a linear classifier in the feature space:

$$LC_\phi(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^N (\phi_j(g))^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (5)$$

for every  $g \in \mathcal{L}$ , with  $\beta'_j \in \mathbb{R}^n$ , for all  $j = 1, \dots, N$ . In what follows, we consider both  $\{\beta'_j\}_{j=1}^N$  and  $\{\omega_j\}_{j=1}^N$  as parameters of  $LC_\phi(\cdot)$ . Finally, we introduce the following definition to simplify notation.

**Definition 4 ( $\Phi$ -separability)** *A pair of sets of gambles  $(A, B)$  is  $\Phi$ -separable, if there exists a classifier  $LC_\phi(\cdot)$  of the type (5), such that  $LC_\phi(A) = 1$  and  $LC_\phi(B) = -1$ . In this case, we indicate the set of these classifiers with  $LC_\Phi(A, B)$ .*

We can now state the main result of this section.

**Proposition 5** *Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise linearly separable, then it is also  $\Phi$ -separable. Vice versa, if there exists a classifier  $LC_\phi(\cdot) \in LC_\Phi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\omega_j = \beta'_j$ , for*

*all  $j = 1, \dots, N$ ,<sup>12</sup> then  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is also piecewise linearly separable.*

The proof of this proposition is based on the observation that:  $\min(g^T \beta_1, \dots, g^T \beta_N) \geq 0$  if and only if  $\sum_{j=1}^N (\mathbb{I}_{\mathcal{B}_j}(g)g)^T \beta_j \geq 0$ , when  $\omega_j = \beta_j$  for all  $j$ . Therefore, there is a one-to-one correspondence between classifiers  $PLC(\cdot) \in PLC(\mathcal{A} \cup T, \mathcal{R} \cup F)$  and classifiers  $LC_\phi(\cdot) \in LC_\Phi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\omega_j = \beta'_j$ , for all  $j = 1, \dots, N$ . Further, in this case, the regions  $\{g \in \mathcal{L} : LC_\phi(g) = 1\}$  correspond to coherent sets  $\mathcal{D}$  such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . Moreover,  $\beta'_j \succeq 0$ , for all  $j$ , analogously to the constraints on  $\beta_j$  that we have for the correspondent classifiers  $PLC(\cdot)$ . Analogous considerations can be made for the feature mappings introduced for the other definitions of *coherence* treated in Section 3.1 and in Section 4.1.

**Example 1** *Consider a coin tossing experiment whose outcomes are h, Heads and t, Tails. A gamble  $g$  in this case, has two components  $g(h) = g_1$  and  $g(t) = g_2$ . If the agent accepts  $g$  then they commit themselves to receive/pay  $g_1$  if the outcome is Heads and  $g_2$  if Tails. Assume that  $\mathcal{A} = \{[-1, 2]^T, [-0.5, 3]^T, [2, -1]^T\}$  and  $\mathcal{R} = \{[-1, 1]^T, [-3, 2]^T, [1, -2]^T\}$ , this could correspond to an imprecise rational agent whose  $\mathcal{D} = \overline{\mathcal{E}(\mathcal{A})}$  is shown in Figure 2.1 (left). We can introduce the partition*

$$\mathcal{B}_1 = \{g : \frac{2}{6}g_1 + \frac{1}{6}g_2 \leq \frac{1}{6}g_1 + \frac{2}{6}g_2\}, \quad (6)$$

$$\mathcal{B}_2 = \{g : \frac{1}{6}g_1 + \frac{2}{6}g_2 \leq \frac{2}{6}g_1 + \frac{1}{6}g_2\}, \quad (7)$$

and verify that the classifier defined as:

$$LC_\phi(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^2 (\mathbb{I}_{\mathcal{B}_j}(g)g)^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (8)$$

for every  $g \in \mathcal{L}$ , with  $\beta'_1 = [\frac{2}{6}, \frac{1}{6}]^T$  and  $\beta'_2 = [\frac{1}{6}, \frac{2}{6}]^T$ , classifies  $\mathcal{A} \cup T$  as 1 and  $\mathcal{R} \cup F$  as -1 and, moreover, the region  $\{g \in \mathcal{L} : LC_\phi(g) = 1\}$  corresponds to  $\overline{\mathcal{E}(\mathcal{A})}$ .<sup>13</sup>

## 3. Convexity

When one considers monetary gambles, the linearity assumption D3 is not very realistic. In this section we consider a weaker form of rationality. We assume D1, D2, D4 and a relaxed version of the axiom D3:

**D3\*** : if  $g, h \in \mathcal{D}$ , then  $tg + (1-t)h \in \mathcal{D}$  for each  $t \in [0, 1]$  [Convexity].

10. We call it *partition* with a little abuse of notation. Indeed, we guarantee only that  $\text{int } \mathcal{B}_j \cap \text{int } \mathcal{B}_k = \emptyset$ , for every  $j, k \in \{1, \dots, N\}$ ,  $j \neq k$ , where  $\text{int } \mathcal{B}_j$  represents the interior of  $\mathcal{B}_j$  in the usual topology of  $\mathbb{R}^n$ . Instead, it is guaranteed that  $\cup_{j=1}^N \mathcal{B}_j = \mathcal{L}$  because every  $g \in \mathcal{L}$  belongs to at least a  $\mathcal{B}_j$ . Indeed, for every  $g \in \mathcal{L}$ ,  $\{g^T \omega_j\}_{j \in N}$  is a finite set of real values, so the minimum always exists.

11. Therefore,  $\mathbb{I}_{\mathcal{B}_j}(g)g = g$  if  $g \in \mathcal{B}_j$  and  $0_n$  otherwise, where  $0_n$  is the null vector in  $\mathbb{R}^n$ . Analogous notation is used for the other feature mappings described in the article.

12. It is only a sufficient condition. Note also that the feature mapping defined in this section is not unique. We can find other feature mappings that guarantee the same properties (see Example 1).

13. Note that this is not the only feature mapping that we can use to linearly separate  $\mathcal{A} \cup T$  from  $\mathcal{R} \cup F$ . Given the fact that  $\mathcal{D}$  is a convex cone in 2D indeed, there always exist a linear classifier that classifies  $\mathcal{A} \cup T$  as 1 and  $\mathcal{R} \cup F$  as -1 in the feature space determined by the feature mapping  $\eta : g \rightarrow [g_2, \mathbb{I}_{g_1 < 0}(g)g_1, \mathbb{I}_{g_1 \geq 0}(g)g_1]^T$ . In this case, it is sufficient to consider  $\beta' = [1, 2, \frac{1}{2}]^T$ .

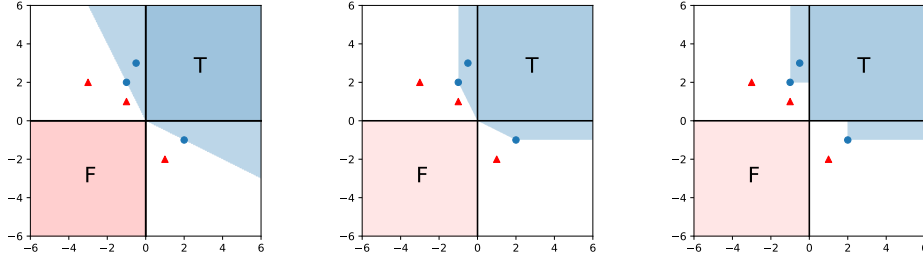


Figure 2.1: Example of a standard (imprecise) coherent set of gambles (left); a *convex* coherence set (center) and a *positive additive* coherent set (right). Points denote gambles in  $\mathcal{A}$ ; triangles gambles in  $\mathcal{R}$ . The blue shaded region is the deductive closure of  $\mathcal{A}$  for the three different notions of coherence.

We define a set  $\mathcal{D}$  that satisfies D1, D2, D3\*, D4 a *convex coherent* set of gambles.

Again, if an agent that is willing to accept and reject respectively a finite set  $\mathcal{A}$  and a finite set  $\mathcal{R}$ , is *rational* in this sense, there exists a convex coherent set of gambles  $\mathcal{D}$ , such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . If this is the case, the minimal set of assumptions that the modeller can make on the agent's beliefs is represented by the least committal convex coherent set that contains  $\mathcal{A}$ . It can be proven that it corresponds to  $\text{ch}(\mathcal{A} \cup T)$ , the *closed convex hull* of  $\mathcal{A} \cup T$  in the usual topology of  $\mathbb{R}^n$  or, equivalently, in the supremum norm topology (see Lemma 23). Furthermore, geometrically, it corresponds to a convex polyhedron (see Lemma 24). Clearly, also in this case, it corresponds to a deductive closure for the set  $\mathcal{A}$  [15, Page 2039].

We claim that proving if an agent is rational in this sense is again equivalent to solve a binary classification task.

**Definition 6 (Binary piecewise affine classifier)** We denote with the term *binary piecewise affine classifier* a classifier  $\text{PAC}(\cdot)$  defined on  $\mathcal{L}$  and characterized by the following discriminant function:

$$\text{PAC}(g) := \begin{cases} 1 & \text{if } g^T \beta_j + \alpha_j \geq 0, \text{ for all } j \in \{1, \dots, N\} \\ -1 & \text{otherwise.} \end{cases} \quad (9)$$

for every  $g \in \mathcal{L}$ , with  $\beta_j \in \mathbb{R}^n$ ,  $\alpha_j \in \mathbb{R}$  for all  $j$ ,  $N \geq 1$ .

**Definition 7 (Piecewise affine separability)** A pair of sets of gambles  $(A, B)$  is *piecewise affine separable* if there exists a binary piecewise affine classifier  $\text{PAC}(\cdot)$ , such that  $\text{PAC}(A) = 1$  and  $\text{PAC}(B) = -1$ . In this case, we indicate the set of these classifiers with  $\text{PAC}(A, B)$ .

Now we can state the main result of this section.

**Proposition 8** Given a pair of finite sets of gambles  $(\mathcal{A}, \mathcal{R})$ , there exists a convex coherent set of gambles  $\mathcal{D}$ , such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ , if and only if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is *piecewise affine separable*.

Analogously as before, from the proof of this proposition, it follows that if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is *piecewise affine separable* then  $\mathcal{D} := \{g \in \mathcal{L} : \text{PLC}(g) = 1\}$  for every  $\text{PLC}(\cdot) \in \text{PLC}(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , is a convex coherent set of gambles such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . Vice versa is not true in general, but it is true for  $\text{ch}(\mathcal{A} \cup T)$ .

It is easy to show moreover that, again similarly as before, every binary piecewise affine classifier that classifies  $T$  as 1 and  $F$  as  $-1$  is characterized by a set of weights  $\{\beta_j, \alpha_j\}_{j=1}^N$  such that  $\beta_j \succeq 0, \alpha_j \geq 0$ , for all  $j = 1, \dots, N$  with at least an  $\alpha_k = 0, k \in \{1, \dots, N\}$ .

### 3.1. Feature Mapping

We can reformulate the previous problem as a linear classification task in a higher dimensional space, using a feature mapping similar to the one seen in Section 2.1. We can indeed define new partitions:

$$\mathcal{B}'_j := \{g' \in \mathcal{L}(\Omega') : g'^T \omega'_j \leq g'^T \omega'_k, \quad (10) \\ \text{for } k = 1, \dots, N, j \neq k\}$$

with  $|\Omega'| = n + 1, \omega'_j \in \mathbb{R}^{n+1}$ .

We can then introduce, the feature mapping  $\psi := (\psi_1, \dots, \psi_N)$ , with  $\psi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ , defined as  $\psi_j : g \rightarrow I_{\mathcal{B}'_j} \left( \begin{bmatrix} g \\ 1 \end{bmatrix} \right) \begin{bmatrix} g \\ 1 \end{bmatrix}$  for every  $j = 1, \dots, N$ . Further, we define the following classifier, which corresponds to a linear classifier in the feature space:

$$\text{LC}_\psi(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^N (\psi_j(g))^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (11)$$

for every  $g \in \mathcal{L}$ , with  $\beta'_j \in \mathbb{R}^{n+1}$ , for all  $j = 1, \dots, N$ . We consider both  $\{\beta'_j\}_{j=1}^N$  and  $\{\omega'_j\}_{j=1}^N$  as parameters of  $\text{LC}_\psi(\cdot)$ . Then we can introduce the following definition.

**Definition 9 ( $\Psi$ -separability)** A pair of sets of gambles  $(A, B)$  is  $\Psi$ -separable, if there exists a classifier  $\text{LC}_\psi(\cdot)$  of the type (11), such that  $\text{LC}_\psi(A) = 1$  and  $\text{LC}_\psi(B) = -1$ . In this case, we indicate the set of these classifiers with  $\text{LC}_\Psi(A, B)$ .

We can now state the main result of this section.

**Proposition 10** *Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is piecewise affine separable, then it is also  $\Psi$ -separable. Vice versa, if there exists a classifier  $LC_\Psi(\cdot) \in LC_\Psi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\omega'_j = \beta'_j$ , for all  $j = 1, \dots, N$ , then  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is also piecewise affine separable.*

The proof of Prop. 10 is similar to that of Prop. 5, indeed it is based on an analogous observation. So, in particular, the region  $\{g \in \mathcal{L} : LC_\Psi(g) = 1\}$  of a classifier  $LC_\Psi(\cdot) \in LC_\Psi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\omega'_j = \beta'_j$  for all  $j$ , is, similarly as before, a convex coherent set  $\mathcal{D}$  such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . Moreover, in this case,  $\beta'_j \geq 0$ , for all  $j = 1, \dots, N$ , with  $\beta'_{j,1:n} \geq 0$ ,<sup>14</sup> with at least a  $\beta'_k$ ,  $k \in \{1, \dots, N\}$ , such that  $\beta'_{k,n+1} = 0$ .

**Example 2** *Consider again Example 1. However now, suppose explicitly to work with monetary gambles, whose values represent rewards in thousands of euros. If the agent has limited financial resources, they can set their maximum loss to 1 thousand of euros, for example. In this case, it is no more reasonable thinking that they will respect axiom D3. A more reasonable rationality definition for the agent, is the one that is based on the axioms D1, D2, D3\*, D4. Analogously to Example 1, if we introduce a classifier  $LC_\Psi(\cdot)$  of the form (11), characterized by the parameters  $\omega'_j = \beta'_j = \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix}$  for  $j = 1, \dots, 4$  where values of  $\beta_j, \alpha_j$  for every  $j$ , correspond to the rows of the following matrices:*

$$\beta = \begin{bmatrix} 1 & 0 \\ \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{6} \\ 0 & 1 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

we can verify that  $LC_\Psi(\cdot)$  classifies  $\mathcal{A} \cup T$  as 1 and  $\mathcal{R} \cup F$  as  $-1$ . Moreover, we have  $\{g : LC_\Psi(g) = 1\} = \text{ch}(\mathcal{A} \cup T)$  (see proofs of Prop. 8 and 10). A graphical representation of  $\text{ch}(\mathcal{A} \cup T)$  is showed in 2.1 (center).

## 4. Positive Additive Coherence

We now consider an even weaker relaxation (see Lemma 22) of the linearity axioms D3:

**D3\*\*** if  $f \in \mathcal{D}$ , then  $f + h \in \mathcal{D}$  for each  $h \in T$  [positive additivity].

We call a set  $\mathcal{D}$  that satisfies D1, D2, D3\*\*, D4 a *positive additive coherent* set of gambles. With a reasoning analogous to the previous sections, if, given a pair of finite sets  $(\mathcal{A}, \mathcal{R})$ , there exists a positive additive coherent set of gambles  $\mathcal{D}$ , such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ , the minimal such set is:  $\uparrow(\mathcal{A} \cup \{0\})$  [11], as Lemma 25 proves. It is possible

14.  $\beta'_{j,1:n}$  denotes the vector containing the first  $n$  components of  $\beta'_j$ .

to notice that, geometrically, it corresponds to a union of orthants centered in the elements of  $\mathcal{A} \cup \{0\}$ . Further, also in this case, it is a deductive closure for  $\mathcal{A}$  [2]. As usual, we show that proving if an agent is rational in this sense is equivalent to solve a binary classification task. Let us introduce the following definitions.

**Definition 11 (PWP classifier)** *We denote with the term binary piecewise positive affine (PWP) classifier, a classifier  $PWPC(\cdot)$  defined on  $\mathcal{L}$  and characterized by the following discriminant function:*

$$PWPC(g) := \begin{cases} 1 & \text{if } \exists f^j \in \mathcal{F} \text{ s.t. } g \geq f^j, \\ -1 & \text{otherwise,} \end{cases} \quad (12)$$

for every  $g \in \mathcal{L}$ , where  $\mathcal{F}$  is a finite set of gambles.

**Definition 12 (PWP separability)** *A pair of sets of gambles  $(A, B)$  is piecewise affine positive separable (PWP) if there exists a PWP classifier  $PWPC(\cdot)$ , such that  $PWPC(A) = 1$  and  $PWPC(B) = -1$ . In this case we indicate the set of these classifiers with  $PWPC(A, B)$ .*

Note that, for every  $j$ ,  $\{g \geq f^j\}$  defines an orthant centered at  $f^j$ , whose border can be expressed as a piecewise affine function. It can easily be proved (by induction on the elements of  $\mathcal{F}$ ) that the decision boundary of (12) is also a piecewise affine function. We can now state the main result of this section.

**Proposition 13** *Given a pair of finite sets of gambles  $(\mathcal{A}, \mathcal{R})$ , there exists a positive additive coherent set  $\mathcal{D}$ , such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ , if and only if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is PWP separable.*

Also here, if  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is PWP separable then  $\mathcal{D} := \{g \in \mathcal{L} : PWPC(g) = 1\}$  for every  $PWPC(\cdot) \in PWPC(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , is a positive additive coherent set such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ . Vice versa is not true in general, but it is true for  $\uparrow(\mathcal{A} \cup \{0\})$ .

### 4.1. Feature Mapping

In this section, we will prove that, also in this case, the previous problem can be reformulated as a linear classification task in a higher dimensional space. Let  $\{\zeta_{i,j}\}$  with  $i = 1, \dots, n$ ,  $j = 1, \dots, N$  denote another partition of  $\mathcal{L} = \mathbb{R}^n$  defined as follows:

$$\zeta_{i,j} := \{g \in \mathcal{L}(\Omega) : (g_i - \omega_i^j) = \max_k (\min_l (g_l - \omega_l^k))\} \quad (13)$$

with  $\omega^j \in \mathbb{R}^n$ ,  $j = 1, \dots, N$ . We can introduce the feature mapping  $\rho := (\rho_1, \dots, \rho_N)$ , with  $\rho_j : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$  defined as:

$$\rho_{j,i}(g) := \begin{cases} \mathbb{I}_{\zeta_{i,j}}(g)g_i & \text{if } i = 1, \dots, n \\ \mathbb{I}_{\zeta_{i-n,j}}(g) & \text{otherwise} \end{cases}$$

for every  $g \in \mathcal{L}$  and  $j = 1, \dots, N$ . Further we define the following classifier, which corresponds to a linear classifier in the feature space:

$$LC_\rho(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^N (\rho_j(g))^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (14)$$

for every  $g \in \mathcal{L}$ , with  $\beta'_j \in \mathbb{R}^{2n}$ . We consider both  $\{\beta'_j\}_{j=1}^N$  and  $\{\omega^j\}_{j=1}^N$  as parameters of  $LC_\rho(\cdot)$ . Similarly as before, we can introduce the following definition.

**Definition 14 (P-separability)** A pair of sets of gambles  $(A, B)$  is P-separable, if there exists a classifier  $LC_\rho(\cdot)$  of the type (14), such that  $LC_\rho(A) = 1$  and  $LC_\rho(B) = -1$ . In this case, we indicate the set of these classifiers as  $LC_P(A, B)$ .

We can now state the main result of this section.

**Proposition 15** Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is PWP separable, then it is also P-separable. Vice versa, if there exists a classifier  $LC_\rho(\cdot) \in LC_P(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\beta'_{j,i} > 0$  and  $\omega_i^j = -\frac{\beta'_{j,i+n}}{\beta'_{j,i}}$ <sup>15</sup> for all  $i, j$ , then  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is also PWP separable.

The proof is based on the following observation, analogous to the previous ones:

$$g \in \{PWPC(g) = 1\} \iff \max_j (\min_i (g_i - f_i^j)) \geq 0 \\ \iff \sum_{j=1}^N \left( \begin{bmatrix} \mathbb{I}_{\zeta_{1,j}}(g) g_1 \\ \vdots \\ \mathbb{I}_{\zeta_{n,j}}(g) g_n \\ \mathbb{I}_{\zeta_{1,j}}(g) \\ \vdots \\ \mathbb{I}_{\zeta_{n,j}}(g) \end{bmatrix} \right)^T \begin{bmatrix} 1 \\ \vdots \\ -f_1^j \\ \vdots \\ -f_n^j \end{bmatrix} \geq 0.$$

if  $\zeta_{i,j} = \{g \in \mathcal{L} : (g_i - f_i^j) = \max_k (\min_l (g_l - f_l^k))\}$ , for all  $i, j$ , where  $f^j$  are the parameters of the PWP classifier  $PWPC(\cdot)$ . Also here the region  $\{g \in \mathcal{L} : LC_\rho(g) = 1\}$  of a classifier  $LC_\rho(\cdot) \in LC_P(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\beta'_{j,i} > 0$  and  $\omega_i^j = -\frac{\beta'_{j,i+n}}{\beta'_{j,i}}$  for every  $i, j$ , is, similarly as before, a positive additive coherent set  $\mathcal{D}$  such that  $\mathcal{D} \supseteq \mathcal{A}$  and  $\mathcal{D} \cap \mathcal{R} = \emptyset$ .

**Example 3** Consider again Example 1, but in this case we assume a positive additive rational agent, i.e. an agent who respects rationality axioms D1, D2, D3\*\*, D4 (but not necessarily D3 or D3\*). Analogously to Example 1 and 2, if we introduce a classifier  $LC_\rho(\cdot)$  of the form (14), characterized by the parameters  $\omega_i^j = -\beta'_{j,i+n}/\beta'_{j,i}$ , and  $\beta'_j$ ,  $i = 1, 2$ ,  $j = 1, \dots, 3$  defined in the following way:

$$\beta'_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \beta'_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \beta'_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

15. These constraints are not so restrictive. Indeed, it is computationally simple to verify if there exists such a classifier in this situation. It is sufficient to find a PWP classifier with  $\mathcal{F} \subseteq (\mathcal{A} \cup \{0\})$  that classifies  $\mathcal{A} \cup T$  as 1 and  $\mathcal{R} \cup F$  as -1.

it classifies  $\mathcal{A} \cup T$  as 1 and  $\mathcal{R} \cup F$  as -1.<sup>16</sup> Moreover,  $\{g : LC_\rho(g) = 1\} = \uparrow (\mathcal{A} \cup \{0\})$  (see proofs of Prop. 13 and 15).

## 5. Lower Previsions and Preferences

Consider a generic feature map  $\chi = (\chi_1, \dots, \chi_N)$ , with  $\chi : \mathbb{R}^n \mapsto \mathbb{R}^M$  with  $M \geq n$ . From every classifier  $LC_\chi(\cdot)$  defined as:

$$LC_\chi(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^N (\chi_j(g))^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (15)$$

for every  $g \in \mathcal{L}$ , with  $\beta'_j \in \mathbb{R}^M$ , for all  $j = 1, \dots, N$  we can induce a preference relation on gambles of the original space  $\mathbb{R}^n$  and therefore a lower and an upper prevision.

**Definition 16** Given a classifier  $LC_\chi(\cdot)$  defined as in (15), we say that  $f \succsim_{LC_\chi} g$  with  $f, g \in \mathbb{R}^n$ , if  $LC_\chi(f - g) = 1$ . In this case, for every gamble  $g \in \mathbb{R}^n$ , we call the following values:

$$\underline{P}_{LC_\chi}(g) := \sup\{c \in \mathbb{R} : LC_\chi(g - c) = 1\}, \\ \bar{P}_{LC_\chi}(g) := \inf\{c \in \mathbb{R} : LC_\chi(c - g) = 1\},$$

respectively the generalized lower and the generalized upper prevision of  $g$ .

Let us consider the case in which  $\chi$  coincides with a feature mapping  $\phi$  defined as in Section 2.1. Given a  $\Phi$ -separable pair of sets  $(\mathcal{A}, \mathcal{R})$ , from every classifier  $LC_\phi(\cdot) \in LC_\Phi(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , we can induce a preference relation, and consequently a generalized lower and a upper prevision, on  $\mathbb{R}^n$ . In particular, the following result is valid.

**Proposition 17** Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If there exists a classifier  $LC_\phi(\cdot) \in LC_\Phi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\omega_j = \beta'_j$ , for all  $j = 1, \dots, N$ , then there exists a coherent set of gambles  $\mathcal{D}$  (i.e. a set of gambles satisfying D1, D2, D3, D4), such that:

$$f \succsim_{LC_\phi} g \iff f - g \in \mathcal{D}, \quad (16)$$

for every  $f, g \in \mathcal{L}$ . Moreover, if  $f - g \in (\mathcal{A} \cup T)$ , then  $f \succsim_{LC_\phi} g$  and if  $f - g \in (\mathcal{R} \cup F)$ , then  $(f \not\succsim_{LC_\phi} g)$ . In particular, it follows that for every  $g \in \mathcal{L}$ :

$$\underline{P}_{LC_\phi}(g) = \underline{P}(g), \quad (17)$$

where  $\underline{P}(g) := \sup\{c \in \mathbb{R} : g - c \in \mathcal{D}\}$  for every  $g \in \mathcal{L}$ , is the standard coherent lower prevision of  $g$  associated to  $\mathcal{D}$ .

The proof follows immediately from the proofs of Prop. 3 and 5. It follows then that the preference relation  $\succsim_{LC_\phi}$ , in this case, satisfies the following properties.

16. The coefficients  $\beta'_j$  of (12) can be determined by solving a simple mixed-integer linear programming problem.

- R1.  $-1 \succsim_{LC_\phi} 0$  [Avoiding Sure Loss];
- R2. If  $f \geq g$  then  $f \succsim_{LC_\phi} g$ , for any  $f, g \in \mathcal{L}(\Omega)$  [Monotonicity];
- R3. If  $f \succsim_{LC_\phi} g$  and  $\lambda > 0$  then  $\lambda f \succsim_{LC_\phi} \lambda g$ , for any  $f, g \in \mathcal{L}(\Omega)$  [Positive homogeneity];
- R4. If  $f \succsim_{LC_\phi} g$  and  $g \succsim_{LC_\phi} h$  then  $f \succsim_{LC_\phi} h$ , for any  $f, g, h \in \mathcal{L}(\Omega)$  [Transitivity];
- R5. If  $f + \delta \succsim_{LC_\phi} g$  for every  $\delta > 0$  then  $f \succsim_{LC_\phi} g$ , for any  $f, g \in \mathcal{L}(\Omega)$  [Continuity];
- R6.  $f \succsim_{LC_\phi} g$  if and only if  $f - g \succsim_{LC_\phi} 0$ , for any  $f, g \in \mathcal{L}(\Omega)$  [Cancellation].

We can prove similar results for the other cases.

**Proposition 18** *Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If there exists a classifier  $LC_\psi(\cdot) \in LC_\Psi(\mathcal{A} \cup T, \mathcal{R} \cup F)$  defined as in (11) with  $\omega'_j = \beta'_j$ , for all  $j = 1, \dots, N$ , then there exists a convex coherent set of gambles  $\mathcal{D}$  (i.e. a set of gambles satisfying D1, D2, D3\*, D4), such that:*

$$f \succsim_{LC_\psi} g \iff f - g \in \mathcal{D}, \quad (18)$$

for every  $f, g \in \mathcal{L}$ . Moreover, if  $f - g \in (\mathcal{A} \cup T)$ , then  $f \succsim_{LC_\psi} g$  and if  $f - g \in (\mathcal{R} \cup F)$ , then  $f \not\succsim_{LC_\psi} g$ .

In this case, R1, R2, R5, R6 remains satisfied. Instead, R3 and R4 can be replaced by:

$$(f \succsim_{LC_\psi} g, h \succsim_{LC_\psi} l) \Rightarrow \alpha f + (1 - \alpha)h \succsim_{LC_\psi} \alpha g + (1 - \alpha)l$$

for any  $\alpha \in [0, 1]$ ,  $f, g, h, l \in \mathcal{L}(\Omega)$ ,

that implies, in particular, the convexity axiom of [17].

**Proposition 19** *Let  $(\mathcal{A}, \mathcal{R})$  be a pair of finite sets of gambles. If there exists a classifier  $LC_\rho(\cdot) \in LC_P(\mathcal{A} \cup T, \mathcal{R} \cup F)$  with  $\beta'_{j,i} > 0$  and  $\omega'_i = \frac{\beta'_{i,i+n}}{\beta'_{j,i}}$ , for all  $i = 1, \dots, n$ ,  $j = 1, \dots, N$ , then there exists a positive additive coherent set of gambles  $\mathcal{D}$  (i.e. a set of gambles satisfying D1, D2, D3\*\*, D4), such that:*

$$f \succsim_{LC_\rho} g \iff f - g \in \mathcal{D}, \quad (19)$$

for every  $f, g \in \mathcal{L}$ . Moreover, if  $f - g \in (\mathcal{A} \cup T)$ , then  $f \succsim_{LC_\rho} g$  and if  $f - g \in (\mathcal{R} \cup F)$ , then  $f \not\succsim_{LC_\rho} g$ .

Also in this case, R1, R2, R5, R6 are still satisfied.

**Example 4** *Consider again Example 1. As we have seen, the pair of sets  $(\mathcal{A}, \mathcal{R})$  is  $\Phi$ -,  $\Psi$ - and  $P$ -separable. The three classifiers  $LC_\phi(\cdot), LC_\psi(\cdot), LC_\rho(\cdot)$  considered in the previous examples moreover, respect the properties required by Props. 17, 18 and 19, respectively. In particular, they induce different preferences relations and different generalized lower previsions on gambles.*

*In the following table are summarized the values of the three generalized lower previsions for  $g = [2, -1]^T$ ,  $g' =$*

*$0.5g$  and  $g'' = 2g$ . From these values, it is possible to notice that  $P_{LC_\psi}(\cdot)$  and  $P_{LC_\rho}(\cdot)$  are not coherent lower previsions (w.r.t. the standard coherence axioms for lower previsions).  $P_{LC_\phi}(\cdot)$  instead, coincides with a standard coherent lower prevision, as proven in Prop. 17.*

	$P_{LC_\phi}(\cdot)$	$P_{LC_\psi}(\cdot)$	$P_{LC_\rho}(\cdot)$
$g$	0	0	0
$g' = 0.5g$	0	0	-0.5
$g'' = 2g$	0	-1	-1

## 6. General Nonlinear Classifier

One may wonder what new kinds of applications are possible once we have redefined desirability as a classification problem. Here we provide the intuition for a general approach to define a nonlinear ‘consequence operator’ via a nonlinear classifier. This would allow us to perform inferences (computing lower previsions and preferences), from a pair of sets  $(\mathcal{A}, \mathcal{R})$ , when we do not know the ‘type of rationality’ of the agent (i.e. we only assume D1, D2 and D4). To this purpose, we introduce a general feature mapping  $\hat{\phi} = (\hat{\phi}_1, \dots, \hat{\phi}_N)$ . Its  $j$ -th component  $\hat{\phi}_j : \mathbb{R}^n \mapsto \mathbb{R}^M$  with  $M \geq n$ , is defined as:

$$\hat{\phi}_{ji}(g) := \begin{cases} \hat{\psi}_{ji}(g)g_i & \text{if } i \leq n, \\ \hat{\psi}_{ji}(g) & \text{otherwise.} \end{cases}$$

for every  $g \in \mathcal{L}$ . We assume that  $\hat{\psi}_{ji}$  for every  $i, j$ , is a scalar function satisfying: (i) for every  $g$ ,  $\hat{\psi}_{ji}(g) \geq 0$ , for all  $ji$  and exists  $kl$  with  $l \leq n$ , such that  $\hat{\psi}_{kl}(g) > 0$ ; (ii) it depends on a vector of parameters  $\theta \in \mathbb{R}^p$ ,  $p \geq 1$ , for all  $j, i$ . We then define the following classifier, which corresponds to a linear classifier in the feature space:

$$LC_{\hat{\phi}}(g) := \begin{cases} 1 & \text{if } \sum_{j=1}^N (\hat{\phi}_j(g))^T \beta'_j \geq 0 \\ -1 & \text{otherwise.} \end{cases} \quad (20)$$

As usual, we consider both  $\{\beta'_j\}_{j=1}^N$  and  $\theta$  as parameters of  $LC_{\hat{\phi}}(\cdot)$ . We introduce then the following definition.

**Definition 20 ( $\hat{\Phi}$ -separability)** *A pair of sets of gambles  $(A, B)$  is  $\hat{\Phi}$ -separable if there exist a classifier  $LC_{\hat{\phi}}(\cdot)$  of the type (20), such that  $LC_{\hat{\phi}}(A) = 1$  and  $LC_{\hat{\phi}}(B) = -1$ . In this case we indicate the set of these classifiers with  $LC_{\hat{\Phi}}(A, B)$ .*

It is easy to see that, depending on the form of  $\hat{\psi}$ ,  $\hat{\Phi}$ -separability becomes  $\Phi$ -separability,  $\Psi$ -separability or  $P$ -separability. However, in the previous sections we have seen that  $\Phi, \Psi, P$ -separability of a pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$ , where  $\mathcal{A}$  and  $\mathcal{R}$  are finite sets of gambles, is not sufficient to guarantee some form of rationality of the agent who is



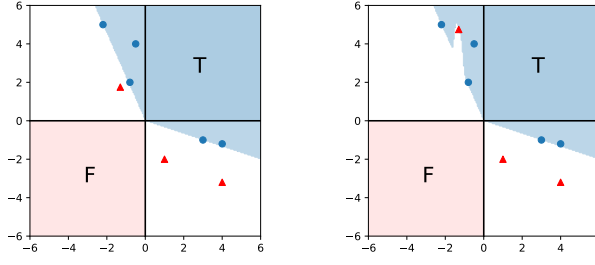


Figure 5.1: Nonlinear classifier for piecewise linearly separable (left) and for non piecewise linearly separable (right)  $\mathcal{A}, \mathcal{R}$ .

willing to accept  $\mathcal{A}$  and reject  $\mathcal{R}$  (see Prop. 5, 10, 15). Nevertheless, it is possible to see that  $\mathcal{D} = \{g : LC_{\hat{\phi}}(g) = 1\}$ , for every  $LC_{\hat{\phi}}(\cdot) \in LC_{\hat{\phi}}(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is such that  $\mathcal{D} \supseteq \mathcal{A}$ ,  $\mathcal{D} \cap \mathcal{R} = \emptyset$  and it satisfies D1, D2, D4.<sup>17</sup> Further, with this general family of feature mappings, we are potentially able to linearly separate in the feature space almost any pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$ . So, it can be useful to constrain the possible feature mappings in such a way that the classifier, when its parameters are appropriately selected, is able to ‘adapt’ itself to the rationality of the subject. If, for example,  $(\mathcal{A}, \mathcal{R})$  is compatible with a stricter notion of coherence (for instance the pair  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  is also piecewise linearly separable), then we would like that our classifier, when optimized, collapses to a classifier  $LC_{\hat{\phi}}(\cdot)$  defined in (5). We provide an example hereafter to clarify our intent.

**Example 5** Consider a possibility space of size  $n = 2$ . Then, every gamble  $g$  has two components  $g_1$  and  $g_2$ . Consider the feature mapping  $\hat{\phi}(\cdot)$  whose non-null components are collected in the following vector:

$$\begin{aligned} & [g_2, \mathbb{I}_{g_1 < 0}(g)g_1, \mathbb{I}_{g_1 \geq 0}(g)g_1, \\ & \exp\left(\frac{-(g_1 - r_1)^2}{\sigma_1^2}\right)g_1, \dots, \exp\left(\frac{-(g_1 - r_{|\mathcal{R}|})^2}{\sigma_{|\mathcal{R}|}^2}\right)g_1], \end{aligned}$$

where  $r_i$  denotes the first component of the  $i$ -th gamble in  $\mathcal{R}$  for  $i = 1, \dots, |\mathcal{R}|$ . There are  $|\mathcal{R}|$  parameters,  $\theta_i = \sigma_i^2$ , which are the variances of the ‘Gaussian’ densities. We constrain  $\beta' \succeq 0$  so that  $LC_{\hat{\phi}}(T) = 1$  and  $LC_{\hat{\phi}}(F) = -1$ . Among the classifiers that satisfy  $LC_{\hat{\phi}}(\mathcal{A}) = 1$  and  $LC_{\hat{\phi}}(\mathcal{R}) = -1$ , we select the one which minimises the objective function

$$\sum_{g_j \in \mathcal{A}} \left( \sum_{i=1}^{|\mathcal{R}|+3} \hat{\phi}_i(g_j) \beta'_i \right) + \gamma \sum_{i=4}^{|\mathcal{R}|+3} \beta'_i, \quad (21)$$

with  $\gamma \geq 0$ . Note that  $\sum_{i=1}^{|\mathcal{R}|+3} \hat{\phi}_i(g_j) \beta'_i \geq 0$  for  $g_j \in \mathcal{A}$  (coherence constraint). Therefore, by minimizing the first

17. Thanks to the assumptions required on  $\hat{\psi}$ , the problem boils down to classifying only  $\mathcal{A}$  and  $\mathcal{R}$  providing the coefficients  $\beta'_i$  satisfy some constraints similar to the ones seen in the previous sections.

term in the objective function, we are minimizing the sum of the distances<sup>18</sup> between the points  $g_j \in \mathcal{A}$  and the hyperplane  $\sum_{i=1}^{|\mathcal{R}|+3} \hat{\phi}_i(g) \beta'_i = 0$ . This means we are looking for the smallest extension in the feature space: the smallest set compatible with the assessments  $\mathcal{A}$ . The second term penalises the coefficients  $\beta'_i$  of the terms  $\hat{\phi}_i(\cdot)$  involving the gaussian densities. This forces the classifier to be piecewise linear when  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  are piecewise linearly separable. The above nonlinear nonconvex optimisation problem can be solved numerically. We provide two examples using  $\gamma = 100$  and  $\mathcal{A} = \{(-0.5, 4), (-2.2, 5), (-0.8, 2.0), (3, -1), (4, -1.2)\}$ . Figure 5.1 (left) shows a case where  $\mathcal{R} = \{(-1.3, 1.75), (1, -2), (4, -3.2)\}$  and so  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  are piecewise linearly separable. The solution of (21) gives  $\beta'_1 = 1, \beta'_2 = 2.272, \beta'_3 = 0.33, \beta'_4 = \beta'_5 = \beta'_6 = 0$ . In Figure 5.1 (right)  $\mathcal{R} = \{(-1.3, 4.75), (1, -2), (4, -3.2)\}$  and so  $(\mathcal{A} \cup T, \mathcal{R} \cup F)$  are not piecewise linearly separable. The solution of (21) gives  $\beta'_1 = 1, \beta'_2 = 2.272, \beta'_3 = 0.33, \beta'_4 = 3.45, \beta'_5 = \beta'_6 = 0$  and  $\sigma_1^2 = 0.038$ . The lower prevision of the gamble  $(-0.5, 4)$  is 0.87 in the first case, and 0.54 in the second case.

## 7. Conclusions

We provided a new interpretation of (nonlinear) desirability as a classification problem. We have considered three instances of nonlinear desirability – imprecision, convex coherence and positive additive coherence – and showed that they can be expressed within this general framework. There are several research directions we aim to pursue in future works: (i) minimum coherence desiderata for the feature mapping; (ii) marginalisation and conditioning. Besides, given the connection between desirability and Von Neumann-Morgenstern’s axiomatization of rationality [18, 32, 33], we plan to investigate if this general framework allows us to represent non-expected utility theory.

18. Technically, the distance is  $(\sum_{i=1}^{|\mathcal{R}|+3} \hat{\phi}_i(g) \beta'_i) / \|\beta'\|$ , but w.l.o.g. we can rescale  $\beta'_i / \|\beta'\|$ .

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